Distributionally Fair Bilevel Optimization

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Abstract

We study distributional fairness in stochastic hierarchical decision making. A leader chooses a decision before uncertainty is observed, a follower subsequently optimizes a scenario-dependent response, and the resulting decisions induce random utilities for multiple communities. We measure unfairness by the maximum pairwise squared Wasserstein distance between the communities' utility distributions and minimize this measure subject to an upper bound on the leader's expected cost. This formulation captures fairness beyond expected outcomes while recognizing that the leader can influence, but cannot directly choose, the follower's actions. Using sample average approximation, we develop two complementary solution approaches. First, we derive an exact mixed-integer nonlinear reformulation based on empirical quantiles and lower-level optimality conditions. Second, under difference-of-convex (DC) representability of the model functions and lower-level value function, we obtain an equivalent DC-constrained formulation and propose an inexact DC algorithm for computing a nearly ϵ-KKT point with an explicit iteration-complexity guarantee. We also show how the value functions and their subgradients can be computed efficiently in two motivating applications: capacity planning and a newsvendor pricing game. Computational experiments demonstrate that the continuous method finds high-quality solutions, often within a few minutes and with small gaps relative to available lower bounds, whereas the exact Gurobi formulation frequently reaches a four-hour time limit. The results show that distributional fairness can be incorporated into large stochastic leader–follower models without sacrificing computational practicality.

Keywords: Stochastic programming, Fairness in decision making, Bilevel optimization, Difference-of-convex optimization

1 Introduction

Bilevel optimization is a critical hierarchical decision-making framework arising in various practical settings in which a leader makes an upper-level decision, while a follower responds by determining the optimal solution to a lower-level optimization problem given the leader's decision. However, the data underlying real-world decision-making processes are often subject to inherent randomness, highlighting the need to account for stochasticity within bilevel optimization frameworks. To address this critical problem setting, we consider a stochastic bilevel optimization problem in which a leader first makes a decision $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^p$, after which the random vector $\boldsymbol{\xi}$ which has a probability distribution $\mathbb{P}_{\boldsymbol{\xi}}$ supported on set $\Xi \subset \mathbb{R}^{d_1}$ is revealed. The follower then observes both $\mathbf{x}$ and $\boldsymbol{\xi}$ and chooses a decision $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^q$, subject to the r constraints $\mathbf{H}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \leq 0$, where $\mathbf{H}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) = (H_1(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}), \ldots, H_r(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}))^\top$. Subsequently, the random vector $\boldsymbol{\zeta}$ which has

a probability distribution $\mathbb{P}_\zeta$ supported on set $\mathrm{Z} \subset \mathbb{R}^{d_2}$ is revealed, and the leader and follower incur costs $G(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\zeta})$ and $Q(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\zeta})$, respectively. We assume $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are independent. [1] This problem can be formulated as the following bilevel optimization problem:

$$
\begin{aligned}
c^* := \min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}(\cdot)} \; & \left\{ \mathbb{E}_{\boldsymbol{\xi}} \left[\mathbb{E}_{\boldsymbol{\zeta}} \left[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) | \boldsymbol{\xi} \right] \right] = \mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\zeta}} \left[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) \right] \right\} \\
\text{s.t. } & \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{ \mathbb{E}_{\boldsymbol{\zeta}} \left[Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}, \boldsymbol{\zeta}) \right] : \mathbf{H}(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) \leq 0 \right\}, \; \forall \boldsymbol{\xi} \in \Xi,
\end{aligned}
\tag{1}
$$

where $\mathcal{X}$ and $\mathcal{Y}$ are closed convex sets, and $\mathbf{y}(\cdot)$ is a decision rule mapping each realization of $\boldsymbol{\xi} \in \Xi$ to a follower decision in $\mathcal{Y}$.

Model (1) minimizes the leader's expected cost while anticipating the follower's optimal response, but does not explicitly control how the resulting benefits and burdens are distributed. The affected individuals may form distinct *communities*, such as geographic service areas, demographic groups, or customers served by different retailers, whose outcomes are not fully captured by the decision makers' objectives and may be distributed inequitably across communities.

These considerations arise in a range of hierarchical decision-making settings. In ride-hailing, a platform can be viewed as the leader, setting pricing, incentive, and information policies, while drivers act as followers through their acceptance and cancellation decisions. Ge et al. (2020) document differences in cancellation risk associated with passenger names conveying different demographic characteristics. Here, communities correspond to passenger groups, and their utilities may measure successful service or cancellations experienced. Fairness therefore depends on both platform policies and drivers' responses. Additionally, in electricity provision, Flores et al. (2023) document demographic disparities in outage severity and duration during the 2021 Texas power crisis. In a bilevel interpretation of emergency electricity allocation, the Electric Reliability Council of Texas (ERCOT) can be modeled as the leader, specifying load-shedding requirements, that is, the amount of electricity demand that must be temporarily disconnected to maintain grid reliability. Transmission operators and local distribution utilities act as followers, selecting service interruptions to meet these requirements subject to local operating constraints (Electric Reliability Council of Texas 2024). Communities correspond to geographic areas or demographic groups, whose utilities can be measured by unmet electricity demand or outage duration. Their burdens therefore depend on both system-level directives and local implementation decisions, motivating fairness considerations across the two decision levels.

Uncertainty introduces an additional dimension to these disparities. Communities with similar expected outcomes may nevertheless face different variability or risks of severe service failures. Comparing expected outcomes alone can therefore overlook substantial differences in their experiences. This motivates incorporating fairness across the distributions of community outcomes into the stochastic bilevel framework (1), accounting for both the leader's decisions and the follower's optimal responses.

To formalize this setting, suppose there are S communities, indexed by $\mathcal{S} = \{1, \ldots, S\}$, For each community $s \in \mathcal{S}$, let $U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})$ denote its utility, abbreviated as U_s whenever the arguments are clear from the context. This utility is stochastic through both its direct dependence on $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ and the follower's response $\mathbf{y}(\boldsymbol{\xi})$ to the realized scenario $\boldsymbol{\xi}$. The leader seeks to promote fairness across communities through its decision $\mathbf{x}$, which affects community utilities directly and indirectly by influencing the follower's optimal response through the lower-level objective and constraints. To quantify the unfairness induced by $\mathbf{x}$ and the decision rule $\mathbf{y}(\cdot)$ through the community utilities, we consider a fairness metric based on the distributions of U_s, $s \in \mathcal{S}$. Specifically, we measure unfairness

[1] The independence assumption is imposed primarily for ease of exposition. Our methodology can be extended to settings in which this assumption does not hold.

by the maximum Wasserstein distance between the utility distributions of any two communities. To introduce this metric formally, we first define the Wasserstein 2-distance between two random vectors.

Definition 1. Given two random vectors ξ_1 and ξ_2 in $\mathbb{R}^d$ with distributions $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively, the Wasserstein 2-distance between their distributions is defined as

$$\mathbb{W}_2(\mathbb{P}_1,\mathbb{P}_2) := \inf_{\mathbb{Q}\in\mathcal{Q}_{\mathbb{P}_1,\mathbb{P}_2}} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \|\xi_1-\xi_2\|_2^2 \, \mathbb{Q}(d\xi_1,d\xi_2) \right)^{1/2} \;\middle|\; \begin{array}{l} \mathcal{Q}_{\mathbb{P}_1,\mathbb{P}_2} \text{ is the set of joint distributions} \\ \text{of } \xi_1 \text{ and } \xi_2 \text{ with marginals } \mathbb{P}_1 \text{ and } \mathbb{P}_2 \end{array} \right\}.$$

We define the unfairness associated with the leader's decision $\mathbf{x}$ and the induced follower response $\mathbf{y}(\cdot)$ as

$$\mathbb{U}(\mathbf{x},\mathbf{y}(\cdot)) := \max_{s,s'\in\mathcal{S}} \mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}), \tag{2}$$

where $\mathbb{P}_{U_s}$ denotes the distribution of U_s for $s\in\mathcal{S}$. The distributionally fair bilevel optimization problem (DFBO) studied in this paper is then formulated as

$$\begin{aligned} \min_{\mathbf{x}\in\mathcal{X},\ \mathbf{y}(\cdot)} \ & \mathbb{U}(\mathbf{x},\mathbf{y}(\cdot)) \\ \text{s.t. } & \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi},\boldsymbol{\zeta})\right] \le c, \\ & \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}'\in\mathcal{Y}} \left\{ \mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x},\mathbf{y}',\boldsymbol{\xi},\boldsymbol{\zeta})\right] : \mathbf{H}(\mathbf{x},\mathbf{y}',\boldsymbol{\xi}) \le 0 \right\}, \ \forall \boldsymbol{\xi}\in\Xi, \end{aligned} \tag{3}$$

where c denotes the maximum expected cost that the leader is willing to incur in order to reduce the unfairness measure $\mathbb{U}(\mathbf{x},\mathbf{y}(\cdot))$. For (3) to be feasible, it is necessary that $c\ge c^*$, where c^* denotes the optimal value of the cost-minimization bilevel problem in (1).

Solving (3) directly is computationally challenging. The distributional-fairness objective is generally nonconvex, and the requirement that the follower's response remain optimal introduces an implicit, typically nonconvex feasible set. These difficulties persist even when the uncertainty distributions are replaced by finite empirical distributions. Moreover, as our numerical experiments demonstrate later, applying commercial global optimization solvers to an exact reformulation can require substantial computational effort as the problem size increases. These challenges motivate the complementary approaches developed in this paper: an exact mixed-integer reformulation for computing globally optimal solutions and bounds, and a scalable continuous method for obtaining high-quality solutions to larger instances.

1.1 Related Literature

Fairness in optimization and stochastic programming. Equity has long been studied in operations research as a criterion that complements system efficiency. Classical models use notions such as max–min, proportional, and α-fairness, or penalize dispersion through inequality indices and ordered outcomes; see Ogryczak et al. (2014) and Karsu and Morton (2015) for reviews. A central question is how much efficiency must be sacrificed to obtain a more equitable allocation. Bertsimas et al. (2011) formalize this loss as the price of fairness, while Bertsimas et al. (2012) characterize efficiency–fairness tradeoffs across families of allocation objectives. These approaches commonly evaluate a deterministic vector of realized or expected utilities, or selected summary statistics of that vector. In stochastic settings, however, two communities can have the same expected utility and still face materially different variability, tail behavior, or service reliability. This motivates comparing the entire distributions of community utilities rather than only their means or worst outcomes.

Wasserstein distance provides a natural way to measure such distributional disparities. It has been used to impose distribution-level parity in fair regression (Chzhen et al. 2020) and to control disparities over selected ranges of quantiles (He et al. 2026). Most closely related to our work, Ye et al. (2024) introduce distributionally fair stochastic optimization, in which the maximum pairwise Wasserstein distance between group-utility distributions is minimized subject to an efficiency requirement. They develop mixed-integer convex representations, lower bounds, and scalable algorithms for a single-stage single-level stochastic optimization model. We adopt the same broad principle of comparing full utility distributions, but consider a bilevel decision environment. Community utilities depend not only on the leader's decision and the realized uncertainty, but also on an optimizing follower's response. Thus, the leader cannot choose the utility distributions directly; they must be induced through decisions that remain optimal for the follower. Our formulation also permits additional uncertainty to be realized after the follower acts.

Bilevel models, applications, and fairness. Bilevel optimization models hierarchical decisions in which a leader anticipates the response of a follower. It has a long history in mathematical programming and has been used in pricing, transportation, network design, resource allocation, and, more recently, machine learning; comprehensive treatments and surveys include Dempe et al. (2015), Dempe and Zemkoho (2020). Machine-learning applications include hyperparameter optimization and meta-learning (Bennett et al. 2008, Franceschi et al. 2018, Rajeswaran et al. 2019), neural architecture search (Liu et al. 2018), reinforcement learning (Hong et al. 2023). A small but growing literature uses bilevel structure to promote fairness directly. Wu et al. (2026) generate synthetic client data at the upper level so that a collaboratively trained model at the lower level produces fairer predictions. Ljubić et al. (2026) use a bilevel formulation to impose route-level fairness while ensuring that each vehicle follows a shortest route for its assigned customers. These studies concern prediction fairness or deterministic workload equity. In contrast, we optimize fairness between stochastic distributions of community outcomes generated by a general leader–follower system.

Continuous first-order methods for bilevel optimization. A large first-order literature studies smooth bilevel problems whose lower-level objective is strongly convex. In that setting the lower-level solution is unique and, under suitable regularity conditions, differentiable with respect to the upper-level variables. This permits computation or approximation of a hypergradient by iterative differentiation, implicit differentiation, or related gradient-estimation procedures (Ghadimi and Wang 2018, Rajeswaran et al. 2019, Grazzi et al. 2020, Ji et al. 2021, Yang et al. 2021). Subsequent work develops faster, stochastic, single-loop, projection-based, or fully first-order variants and sharper oracle-complexity guarantees (Khanduri et al. 2021, Chen et al. 2022b,a, Li et al. 2022, Xiao et al. 2022, Hu et al. 2023, Hong et al. 2023, Kwon et al. 2023, Chen et al. 2025, Jiang et al. 2024). These methods are powerful when the required differentiability and stability of the response map hold. They are not directly tailored to our setting, where the lower-level problem may be constrained and nonsmooth and may admit multiple optimal responses.

Recent research weakens lower-level strong convexity or avoids differentiating through a single-valued response map. This literature uses value functions, penalty functions, averaged multipliers, optimality measures, or selection maps, and often replaces strong convexity by the Polyak–Łojasiewicz condition or other error-bound properties (Liu et al. 2021a, 2023a, 2021b, 2023b, Chen et al. 2024, Liu et al. 2022, Kwon et al., Xiao et al. 2023). In particular, Shen and Chen (2023), Shen et al. (2025) analyze penalty-based bilevel gradient methods under conditions that control distance to the lower-level solution set. Penalty ideas for bilevel and generalized bilevel programs also have a longer history (Ishizuka and Aiyoshi 1992, Marcotte and Zhu 1996, Ye et al. 1997). More recently, Lu

and Mei (2024) develop first-order penalty methods for unconstrained and constrained lower-level problems, and Lu and Mei (2026) use sequential minimax optimization for the constrained case. These works substantially broaden the scope of continuous bilevel algorithms, but their assumptions and stationarity analyses differ from the value-function-based DC framework developed here.

Lower-level constraints create additional difficulties because the follower's multipliers and active set may change with the leader's decision. Existing approaches use KKT systems, duality, primal–dual penalties, and specialized value or gap functions. Ouattara and Aswani (2018) obtain a single-level formulation from Lagrangian duality, while Jiang et al. (2024) and Nie et al. (2025) develop primal–dual or augmented-Lagrangian value-function methods. Yao et al. (2024) introduce a proximal Lagrangian value function, and Yao et al. (2025) use a regularized gap function to accommodate general lower-level constraints without requiring lower-level strong convexity. Other first-order methods focus on linearly constrained lower-level problems (Tsaknakis et al. 2022, Khanduri et al. 2023, Kornowski et al.). Wolfe-duality and KKT reformulations can handle more general constraints (Allende and Still 2013, Li et al. 2023), whereas Gao et al. (2022), Ye et al. (2023b) use lower-level value functions to obtain difference-of-convex (DC) reformulations. Hierarchical variational inequalities provide a related framework for two-level equilibrium problems (Khalafi and Boob 2025, Dvurechensky et al. 2026). In addition, several reformulations of bilevel constraints lead to minimax subproblems, connecting this literature to first-order algorithms for nonconvex–concave or constrained minimax optimization (Lin et al. 2020, Kong and Monteiro 2021, Zhang et al. 2022, 2024, Lu and Mei 2025). Our continuous approach belongs to the value-function branch of this literature: it represents both lower-level optimality and the empirical Wasserstein objective through DC functions and analyzes approximate stationarity for the resulting constrained problem.

Mixed-integer methods for bilevel optimization. A complementary literature develops exact or globally oriented methods based on mixed-integer reformulations. Kleinert et al. (2021) survey branch-and-bound, cutting-plane, and decomposition techniques for continuous and mixed-integer bilevel problems. For linear bilevel programs, classical approaches include specialized branch-and-bound rules (Hansen et al. 1992) and extended KKT formulations (Shi et al. 2005). When the lower-level problem is convex and satisfies appropriate regularity conditions, KKT conditions or strong-duality relations can replace follower optimality, after which complementarity and bilinear terms are represented with binary variables, disjunctive constraints, or convex envelopes (Ye and Zhu 1995, Allende and Still 2013, Ouattara and Aswani 2018). The numerical consequences of alternative formulations, including approaches avoiding big-M constants, are studied by Kleinert and Schmidt (2023). Our exact method combines this reformulation perspective with a quantile representation of empirical Wasserstein distance. The resulting mixed-integer nonlinear formulation simultaneously enforces the optimality of the follower's scenario-dependent decisions and the ordering needed to compare community-utility distributions. This exact formulation is intended for moderate-size instances and provides a benchmark for evaluating the continuous method.

Difference-of-convex (DC) optimization. Our scalable method further connects to DC programming and DCA, a well-established framework for structured nonconvex optimization (Tao and An 1997). Convergence analyses and modern variants include results based on subanalyticity, smoothing, and inexact or boosted updates (Sriperumbudur and Lanckriet 2012, Sun and Sun 2023, Yang et al. 2025). DC algorithms (DCAs) have also been developed specifically for bilevel programs through lower-level value functions (Gao et al. 2022, Ye et al. 2023a). Our formulation requires an additional structural step: both the empirical Wasserstein fairness objective and the constraints encoding the follower's response must be represented in DC form. This yields a continuous method

for large instances together with a nearly ϵ-KKT analysis, while the mixed-integer formulation supplies exact solutions or bounds on smaller instances. Thus, the proposed methodology links distributionally fair stochastic optimization with both major computational traditions in bilevel optimization: exact mixed-integer reformulation and continuous first-order approximation.

1.2 Contributions

Our main contributions are summarized as follows.

- We introduce a distributionally fair bilevel optimization framework for stochastic hierarchical decisions. The leader minimizes the maximum pairwise squared Wasserstein distance between community-utility distributions, subject to an expected-cost budget and the requirement that the follower's scenario-dependent decisions remain optimal. The framework encompasses distributionally fair two-stage stochastic programming as a special case.

- Under empirical distributions, we develop both exact and first-order solution methods. We derive an exact mixed-integer nonlinear programming (MINLP) reformulation of the proposed problem combining a quantile representation of the Wasserstein fairness objective with lower-level optimality conditions and present computationally efficient lower bounds.

- For larger-scale instances, we derive DC reformulation of the proposed problem and propose an inexact DCA, establish its convergence to a nearly ϵ-KKT point, and derive an explicit bound on its overall iteration complexity. Unlike most existing methods, which require smoothness and/or a unique lower-level solution, our DCA framework accommodates nonsmooth lower-level problems with potentially nonunique optimal responses. To the best of our knowledge, this is the first iteration-complexity guarantee for a bilevel optimization problem of this form.

- We evaluate the proposed methods on distributionally fair capacity-planning and newsvendor-pricing problems. Although DCA targets stationary points, it produces high-quality solutions substantially faster than the exact reformulation solved by Gurobi, obtaining near-optimal solutions within a few minutes on several instances for which Gurobi finds no feasible solution within four hours. The experiments further illustrate how leader decisions influence distributional fairness through follower responses and quantify the resulting economic trade-offs.

The remainder of the paper is organized as follows: Section 2 provides motivating applications for the proposed setting, Section 3 and Section 4 introduce reformulations through MINLP and DC representations with their respective properties, Section 5 presents the DCA framework, Section 6 provides computational results, and Section 7 concludes the paper.

2 Motivating Applications

We first show that two-stage stochastic programming is a special case of the proposed bilevel optimization framework in Section 2.1. Then, we present two applications of the DFBO model (3) involving capacity planning and newsvendor-pricing problems in Sections 2.2 and 2.3, respectively.

2.1 Two-Stage Stochastic Programming

In a traditional two-stage stochastic program, a first-stage decision $\mathbf{x} \in \mathcal{X}$ is made before the realization of uncertainty and incurs a cost $g(\mathbf{x})$. After a random scenario $\boldsymbol{\xi} \in \Xi$ is revealed, a

second-stage decision $\mathbf{y} \in \mathcal{Y}$ is chosen based on both $\mathbf{x}$ and $\boldsymbol{\xi}$, subject to the constraints $\mathbf{H}(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \leq 0$, and incurs a cost $Q(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$. The resulting two-stage stochastic problem can be formulated as

$$\min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}(\cdot)} g(\mathbf{x}) + \mathbb{E}_{\boldsymbol{\xi}} \left[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \right] \text{ s.t. } \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq 0, \ \forall \boldsymbol{\xi} \in \Xi, \tag{4}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are closed convex sets, and $\mathbf{y}(\cdot)$ is a decision rule mapping each scenario $\boldsymbol{\xi} \in \Xi$ to a second-stage decision in $\mathcal{Y}$. Then, the two-stage stochastic program in (4) can be equivalently written as the following bilevel optimization problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}(\cdot)} \ & g(\mathbf{x}) + \mathbb{E}_{\boldsymbol{\xi}} \left[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \right] \\ \text{s.t. } & \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{ Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) : \mathbf{H}(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) \leq 0 \right\}, \ \forall \boldsymbol{\xi} \in \Xi. \end{aligned} \tag{5}$$

This formulation is a special case of (1) without the second random variable $\boldsymbol{\zeta}$, where $G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = g(\mathbf{x}) + Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$. Thus, by considering a specific community utility function U_s similar to (3), we can formulate a *distributionally fair two-stage stochastic program* as

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}(\cdot)} \ & \mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot)) \\ \text{s.t. } & g(\mathbf{x}) + \mathbb{E}_{\boldsymbol{\xi}} \left[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \right] \leq c, \\ & \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{ Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) : \mathbf{H}(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) \leq 0 \right\}, \ \forall \boldsymbol{\xi} \in \Xi, \end{aligned} \tag{6}$$

where $\mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot))$ is defined as in (2), with the dependence on $\boldsymbol{\zeta}$ removed.

2.2 Capacity Planning

We next consider a special case of the two-stage stochastic program in (4) over a capacity planning and shipment planning problem under demand uncertainty. Suppose there are n production sites that produce a certain type of product to be shipped to S communities, indexed by $\mathcal{S} = \{1, \ldots, S\}$, where community $s \in \mathcal{S}$ consists of m_s customers. The first-stage decision is $\mathbf{x} = (x_1, \ldots, x_n)^\top \in [0,1]^n$, where x_i represents the fraction of the maximum production capacity maintained at production site i and o_i denotes the maximum production capacity of site i. We assume that maintaining capacity $\mathbf{x}$ incurs a linear cost $\mathbf{g}^\top \mathbf{x}$, where $\mathbf{g} = (g_1, \ldots, g_n)^\top \in \mathbb{R}^n_+$.

After the production capacities are determined, then random demand $\boldsymbol{\xi} = (\boldsymbol{\xi}^s)_{s \in \mathcal{S}}$ is realized, where $\boldsymbol{\xi}^s = (\xi^s_1, \ldots, \xi^s_{m_s})^\top \in \mathbb{R}^{m_s}$ and ξ^s_j corresponds to customer j in community s for each $s \in \mathcal{S}$ and $j = 1, \ldots, m_s$. Subsequently, shipment is planned by minimizing the total shipping cost and the penalty for unmet demand. Let $\mathbf{y}(\boldsymbol{\xi}) = (y^s_{ij}(\boldsymbol{\xi}))_{i=1,\ldots,n;\, j=1,\ldots,m_s;\, s \in \mathcal{S}}$ denote the shipment decisions, where $y^s_{ij}(\boldsymbol{\xi})$ is the quantity shipped from production site i to customer j in community s, and let c^s_{ij} be the corresponding per-unit shipping cost. In addition, let p_s denote the per-unit penalty for unmet demand in community s. The capacity planning problem can then be formulated as the following two-stage stochastic program:

$$\begin{aligned} c^*_{CP} := \min_{\mathbf{x} \in [0,1]^n, \mathbf{y}(\cdot) \geq 0} \ & \mathbf{g}^\top \mathbf{x} + \mathbb{E}_{\boldsymbol{\xi}} \left[\sum_{i=1}^n \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} c^s_{ij} y^s_{ij}(\boldsymbol{\xi}) \right] + \mathbb{E}_{\boldsymbol{\xi}} \left[\sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} p_s \left(\xi^s_j - \sum_{i=1}^n y^s_{ij}(\boldsymbol{\xi}) \right) \right] \\ \text{s.t. } & \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} y^s_{ij}(\boldsymbol{\xi}) \leq x_i o_i, \ \forall i = 1, \ldots, n, \ \forall \boldsymbol{\xi} \in \Xi, \\ & \sum_{i=1}^n y^s_{ij}(\boldsymbol{\xi}) \leq \xi^s_j, \ \forall s \in \mathcal{S}, \ j = 1, \ldots, m_s, \ \forall \boldsymbol{\xi} \in \Xi. \end{aligned} \tag{7}$$

In the fairness-aware setting, the goal is to determine production capacities that balance capacity costs, shipping costs, and unmet demand penalties, while promoting fairness by reducing disparities in the distributions of unmet demand across communities. To this end, we define the utility of community s as the average unmet demand across its customers for this problem as U_s^{CP}:

$$U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) = U_s^{\mathrm{CP}} := U_s^{\mathrm{CP}}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) := \frac{1}{m_s} \sum_{j=1}^{m_s} \left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi}) \right), \tag{8}$$

where a larger value of U_s represents a lower service level for community s. We define $\mathbb{U}^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) = \max_{s,s' \in S}(\mathbb{P}_{U_s^{CP}}, \mathbb{P}_{U_{s'}^{CP}})$, and the resulting distributionally fair capacity planning problem can be then formulated as

$$\begin{aligned}
\min_{\mathbf{x} \in [0,1]^n, \mathbf{y}(\cdot)} \quad & \mathbb{U}^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) \qquad (9) \\
\text{s.t. } & \mathbf{g}^\top \mathbf{x} + \mathbb{E}_{\boldsymbol{\xi}} \left[\sum_{i=1}^{n} \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi}) \right] + \mathbb{E}_{\boldsymbol{\xi}} \left[\sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} p_s \left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi}) \right) \right] \leq c, \\
& \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}' \geq 0} \left\{ \sum_{i=1}^{n} \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} c_{ij}^s y_{ij}'^s + \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} p_s \left(\xi_j^s - \sum_{i=1}^{n} y_{ij}'^s \right) : \right. \\
& \qquad \left. \text{s.t. } \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} y_{ij}'^s \leq x_i o_i, \ \forall i = 1, \ldots, n, \sum_{i=1}^{n} y_{ij}'^s \leq \xi_j^s, \ \forall s \in \mathcal{S}, \ j = 1, \ldots, m_s \right\}, \ \forall \boldsymbol{\xi} \in \Xi.
\end{aligned}$$

2.3 Newsvendor Pricing Game in a Two-Echelon Supply Chain

We next consider a special case of (1) arising from a newsvendor pricing game in a two-echelon supply chain. Suppose there is one wholesaler, who acts as the leader, and n retailers, who act as followers. Each retailer serves exactly one community. The wholesaler incurs a production cost c_i for each unit of product sold to retailer i and sets a wholesale price x_i for retailer i, for $i = 1, \ldots, n$. Before the demand of each community, denoted by $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)^\top$, are realized, retailer i chooses an order quantity y_i from the wholesaler. This decision is made based on the per-unit holding cost h_i, backlog cost b_i, retail price r_i, and wholesale price x_i. The wholesaler's total cost, or negative profit, is then $(\mathbf{c} - \mathbf{x})^\top \mathbf{y} = \sum_{i=1}^{n} (c_i - x_i) y_i$, where the total expected cost across all retailers is

$$\mathbb{E}_{\boldsymbol{\zeta}} \left[\sum_{i=1}^{n} b_i (\zeta_i - y_i)_+ + h_i (y_i - \zeta_i)_+ + x_i y_i - r_i \min\{y_i, \zeta\} \right]$$

which corresponds to the standard newsvendor cost. After rearranging this cost, the wholesaler's decision-making problem can be formulated as the following instance of (1):

$$\begin{aligned}
\min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}} \quad & (\mathbf{c} - \mathbf{x})^\top \mathbf{y} \qquad (10) \\
\text{s.t. } & \mathbf{y} \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{ \mathbb{E}_{\boldsymbol{\zeta}} \left[\sum_{i=1}^{n} b_i (\zeta_i - y_i')_+ + (h_i + r_i)(y_i' - \zeta_i)_+ + (x_i - r_i) y_i' \right] \right\},
\end{aligned}$$

where the feasible sets $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \underline{x}_i \leq x_i \leq \overline{x}_i \ \ i = 1, \ldots, n\}$ and $\mathcal{Y} = \{\mathbf{y} \in \mathbb{R}^n : \underline{y}_i \leq y_i \leq \overline{y}_i \quad i = 1, \ldots, n\}$ correspond to the box constraints with lower and upper bounds on the wholesale prices and order quantities, respectively. Specifically, we set

$$\underline{x}_i = -h_i, \quad \overline{x}_i = b_i + r_i, \quad \underline{y}_i = \min_{\boldsymbol{\zeta} \in \mathrm{Z}} \zeta_i, \quad \overline{y}_i = \max_{\boldsymbol{\zeta} \in \mathrm{Z}} \zeta_i, \quad i = 1, \ldots, n, \tag{11}$$

assuming that Z is compact. When $\underline{x}_i \le x_i \le \overline{x}_i$, the retailer's objective is nonincreasing in y_i below $\underline{y}_i$ and nondecreasing above $\overline{y}_i$, so there exists an optimal retailer decision in $[\underline{y}_i, \overline{y}_i]$. In this instance of (1), the wholesaler and retailer determine their pricing and order decisions before observing the demand $\boldsymbol{\zeta}$. Thus, there is no first-stage random variable $\boldsymbol{\xi}$, and both of the decisions of the leader and follower are here-and-now decisions, resulting the follower decision $\mathbf{y}$ to be deterministic.

In the fairness-aware setting, the goal is to determine wholesale prices that promote fairness by reducing disparities in the distributions of unmet demand across communities, while maintaining a prescribed level of profitability for the wholesaler and accounting for retailers' optimal ordering decisions. We set $S = n$ and identify each retailer's served market with one community, so that $\mathcal{S} = \{1, \ldots, n\}$. We define the utility of community s as its unmet demand for this problem as U_s^{PG}:

$$U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) = U_s^{PG} := U_s^{PG}(\mathbf{x}, \mathbf{y}, \boldsymbol{\zeta}) := (\zeta_s - y_s)_+ , \tag{12}$$

where a larger value of U_s represents a lower service level for community s. We define $\mathbb{U}^{PG}(\mathbf{x}, \mathbf{y}) = \max_{s,s' \in S}(\mathbb{P}_{U_s^{PG}}, \mathbb{P}_{U_{s'}^{PG}})$, and the corresponding distributionally fair newsvendor pricing game can be then formulated as

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X}, \mathbf{y}} \ & \mathbb{U}^{PG}(\mathbf{x}, \mathbf{y}) \\ \text{s.t. } & (\mathbf{c} - \mathbf{x})^\top \mathbf{y} \le c, \\ & \mathbf{y} \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{ \mathbb{E}_{\boldsymbol{\zeta}} \left[\sum_{i=1}^{n} b_i(\zeta_i - y_i')_+ + (h_i + r_i)(y_i' - \zeta_i)_+ + (x_i - r_i) y_i' \right] \right\}. \end{aligned} \tag{13}$$

3 Reformulations through Mixed-Integer Nonlinear Programming

The optimal solution of (3) depends on the distributions of the random scenarios $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, which may be unknown in practice. These distributions are needed to evaluate the expectations $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}$ and $\mathbb{E}_{\boldsymbol{\zeta}}$, as well as the Wasserstein distances in (2). We approximate these quantities using empirical distributions constructed from historical samples. Specifically, suppose we are given samples $\mathcal{D}_{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_i\}_{i=1}^{N_{\boldsymbol{\xi}}}$ and $\mathcal{D}_{\boldsymbol{\zeta}} = \{\boldsymbol{\zeta}_j\}_{j=1}^{N_{\boldsymbol{\zeta}}}$ of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, respectively. Since $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are assumed to be independent, we approximate the joint distribution of $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ by the product of their empirical distributions. Consequently, the expectations in (3) are approximated by $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})\right] \approx \frac{1}{N_{\boldsymbol{\xi}} N_{\boldsymbol{\zeta}}} \sum_{i=1}^{N_{\boldsymbol{\xi}}} \sum_{j=1}^{N_{\boldsymbol{\zeta}}} G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}_i), \boldsymbol{\xi}_i, \boldsymbol{\zeta}_j)$, and, for each fixed realization $\boldsymbol{\xi}_i$, $\mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}_i, \boldsymbol{\zeta})\right] \approx \frac{1}{N_{\boldsymbol{\zeta}}} \sum_{j=1}^{N_{\boldsymbol{\zeta}}} Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}_i, \boldsymbol{\zeta}_j)$. Equivalently, this representation can be viewed as replacing the true distributions of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ with the uniform empirical distributions supported on $\mathcal{D}_{\boldsymbol{\xi}}$ and $\mathcal{D}_{\boldsymbol{\zeta}}$, respectively. *To simplify notation, throughout the remainder of the paper we keep the expectation notation in (3), while interpreting $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ as uniformly distributed over $\mathcal{D}_{\boldsymbol{\xi}}$ and $\mathcal{D}_{\boldsymbol{\zeta}}$, resulting in $\Xi = \mathcal{D}_{\boldsymbol{\xi}}$, $\mathrm{Z} = \mathcal{D}_{\boldsymbol{\zeta}}$ with $N := N_{\boldsymbol{\xi}} N_{\boldsymbol{\zeta}}$.*

After this replacement, U_s follows an empirical distribution for any given $\mathbf{x}$ and $\mathbf{y}(\cdot)$. In particular, let $\mathcal{L} = \{(i,j) : i = 1, \ldots, N_{\boldsymbol{\xi}}, j = 1, \ldots, N_{\boldsymbol{\zeta}}\}$. By using the compact notation $\ell = (i,j) \in \mathcal{L}$, for each community $s \in \mathcal{S}$, define

$$u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) := U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}_i), \boldsymbol{\xi}_i, \boldsymbol{\zeta}_j), \qquad \ell = (i,j) \in \mathcal{L}. \tag{14}$$

Hence, U_s equals $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot))$ with probability $1/N$ for any $\ell \in \mathcal{L}$.

Similarly, the Wasserstein distance $\mathbb{W}_2^2(\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}})$ in (2) is then defined using the same empirical distributions. For each pair of communities $s, s' \in \mathcal{S}$, the empirical squared Wasserstein distance between their utility distributions is given by

$$\mathbb{W}_2^2(\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}) := \min_{\boldsymbol{\pi} \in \Pi} \sum_{\ell \in \mathcal{L}} \sum_{\ell' \in \mathcal{L}} \pi_{\ell\ell'} \left[u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - u_{s',\ell'}(\mathbf{x}, \mathbf{y}(\cdot))\right]^2, \tag{15}$$

where Π denotes the set of transportation plans with uniform marginals:

$$\Pi := \left\{ \boldsymbol{\pi} \in \mathbb{R}_+^{N\times N} : \sum_{\ell'\in\mathcal{L}} \pi_{\ell\ell'} = \frac{1}{N},\ \forall \ell \in \mathcal{L},\quad \sum_{\ell\in\mathcal{L}} \pi_{\ell\ell'} = \frac{1}{N},\ \forall \ell' \in \mathcal{L} \right\}.$$

Thus, after replacing the true distributions by their empirical counterparts, (15) is utilized for the exact discrete optimal transport representation of $\mathbb{W}_2^2(\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}})$ in (2) and (3).

3.1 Single-level Reformulation of DFBO

To obtain a MINLP reformulation of the DFBO model (3), we first introduce an epigraph variable ν leading to the following alternative representation:

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{y}(\cdot),\nu)\in\mathcal{F}} \nu \ \ \textbf{s.t.}\ \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi},\boldsymbol{\zeta})\right] \le c, \\ \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}'\in\mathcal{Y}} \left\{ \mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x},\mathbf{y}',\boldsymbol{\xi},\boldsymbol{\zeta})\right] : \mathbf{H}(\mathbf{x},\mathbf{y}',\boldsymbol{\xi}) \le 0 \right\},\ \forall \boldsymbol{\xi} \in \Xi, \end{aligned} \tag{16}$$

where the set $\mathcal{F}$ denotes the epigraph of the Wasserstein unfairness measure, that is

$$\mathcal{F} = \{(\mathbf{x},\mathbf{y}(\cdot),\nu) \in \mathcal{X}\times\mathcal{Y}^{N_{\boldsymbol{\xi}}}\times\mathbb{R}_+ :\ \mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}) \le \nu,\ \forall s,s' \in \mathcal{S}\}. \tag{17}$$

To represent the set $\mathcal{F}$, we consider an alternative representation of $\mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}})$ provided in Santambrogio (2015) and extend its quantile-based reformulation in Ye et al. (2024) to our problem setting. To this end, we first introduce the cumulative distribution functions of the random utilities $\{U_s\}_{s\in\mathcal{S}}$ as $F_s(t|(\mathbf{x},\mathbf{y}(\cdot))) = \text{Prob}(U_s \le t)$ for $s \in \mathcal{S}$. Correspondingly, we define the inverse distribution functions $F_s^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot))) = \inf\{t : F_s(t|(\mathbf{x},\mathbf{y}(\cdot))) \ge w\}$ for $s \in \mathcal{S}$. Then, we have

$$\mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}) = \int_0^1 |F_s^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot))) - F_{s'}^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot)))|^2\ dw, \tag{18}$$

according to Proposition 2.17 in Santambrogio (2015). Since U_s has a discrete distribution of N scenarios, $F_s^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot)))$ remains constant whenever $w \in (\frac{k-1}{N}, \frac{k}{N}]$ for $k = 1,\ldots,N$ for any fixed $\mathbf{x}$ and $\mathbf{y}(\cdot)$. Therefore, the squared Wasserstein distance in (18) can be further represented as

$$\begin{aligned} \mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}) &= \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left|F_s^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot))) - F_{s'}^{-1}(w|(\mathbf{x},\mathbf{y}(\cdot)))\right|^2\ dw \\ &= \sum_{k=1}^N \frac{1}{N} \left|F_s^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot))) - F_{s'}^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot)))\right|^2 \end{aligned} \tag{19}$$

Note that $F_s^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot)))$ must equal $u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot))$ for some $\ell \in \mathcal{L}$ that is the kth largest one among $\{u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot))\}_{\ell\in\mathcal{L}}$. This relationship thus can be characterized using linear inequalities with additional binary variables and McCormick representation of bilinear terms.

Consider a set $\{(\psi,\kappa,\nu) \in \mathbb{R}\times\{\kappa^L,\kappa^U\}\times[\nu^L,\nu^U] : \psi = \kappa\nu\}$, which is defined by a bilinear term involving a binary variable κ and a bounded continuous variable ν. Its equivalent McCormick linear representation (McCormick 1976) is defined as

$$\text{MC}(\kappa^L,\kappa^U,\nu^L,\nu^U) = \left\{ (\psi,\kappa,\nu) : \begin{array}{c} \psi\in\mathbb{R}, \kappa\in\{\kappa^L,\kappa^U\}, \nu^L \le \nu \le \nu^U, \\ \psi \ge \kappa^L\nu + \kappa\nu^L - \kappa^L\nu^L, \psi \ge \kappa^U\nu + \kappa\nu^U - \kappa^U\nu^U, \\ \psi \le \kappa^U\nu + \kappa\nu^L - \kappa^U\nu^L, \psi \le \kappa\nu^U + \kappa^L\nu - \kappa^L\nu^U \end{array} \right\}.$$

Then, we define the graph of $F_s^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot)))$, namely,

$$\Omega_s(k) = \left\{(\mathbf{x},\mathbf{y}(\cdot),t_{ks}) \in \mathcal{X}\times\mathcal{Y}^{N_{\boldsymbol{\xi}}}\times\mathbb{R} : F_s^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot))) = t_{ks}\right\}$$

for $k = 1,\ldots,N$ and for all $s \in \mathcal{S}$. Using the reformulation technique from Ye et al. (2024), we propose the following equivalent formulation of $\Omega_s(k)$.

Lemma 2. *Let $M_k^s \geq \max_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)\in\mathcal{Y}^{N_{\boldsymbol{\xi}}},\ell\in\mathcal{L}} |u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot))|$ for $k = 1,\ldots,N$ and $s \in \mathcal{S}$. The graph $\Omega_s(k)$ can be represented as*

$$\Omega_s(k) = \left\{(\mathbf{x},\mathbf{y}(\cdot),t_{ks}) \in \mathcal{X}\times\mathcal{Y}^{N_{\boldsymbol{\xi}}}\times\mathbb{R} : \begin{array}{l} \upsilon_{\ell ks}\in\{0,1\}, z_{\ell ks}\in\{0,1\}, \hat{t}_{\ell ks}\in\mathbb{R}, \text{ for } \ell\in\mathcal{L}, \\ \upsilon_{\ell ks}\leq z_{\ell ks}, \text{ for } \ell\in\mathcal{L}, \sum_{\ell\in\mathcal{L}} z_{\ell ks} = k, \sum_{\ell\in\mathcal{L}}\upsilon_{\ell ks} = 1, \\ t_{ks} = \sum_{\ell\in\mathcal{L}}\hat{t}_{\ell ks}, \\ t_{ks}\geq u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot)) - 2M_k^s(1-z_{\ell ks}), \\ t_{ks}\leq u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot)) + 2M_k^s z_{\ell ks}, \\ (\hat{t}_{\ell ks},\upsilon_{\ell ks},u_{s,\ell}(\mathbf{x},\mathbf{y}(\cdot))) \in MC(0,1,-M_k^s,M_k^s). \end{array}\right\}. \quad (20)$$

Proof. See Section A.1 in the Appendix. □

By (19) and Lemma 2, we derive a MINLP reformulation of the set $\mathcal{F}$.

Lemma 3. *The set $\mathcal{F}$ in (17) can be represented as $\mathcal{F}'$, where*

$$\mathcal{F}' = \left\{(\mathbf{x},\mathbf{y}(\cdot),\nu) \in \mathcal{X}\times\mathcal{Y}^{N_{\boldsymbol{\xi}}}\times\mathbb{R}_+ : \begin{array}{l} \frac{1}{N}\sum_{k=1}^{N}(t_{ks}-t_{ks'})^2 \leq \nu, \ \forall s,s'\in\mathcal{S}, \\ (\mathbf{x},\mathbf{y}(\cdot),t_{ks})\in\Omega_s(k), k=1,\ldots,N, \forall s\in\mathcal{S}. \end{array}\right\}. \quad (21)$$

Proof. By Lemma 2, $(\mathbf{x},\mathbf{y}(\cdot),t_{ks}) \in \Omega_s(k)$ if and only if $F_s^{-1}(\frac{k}{N}|(\mathbf{x},\mathbf{y}(\cdot))) = t_{ks}$ for $k = 1,\ldots,N$ and $s \in \mathcal{S}$. The conclusion then follows from Lemma 2. □

Next, we focus on the bilevel structure of the proposed model by reformulating the follower's optimality conditions using the Karush-Kuhn-Tucker (KKT) conditions. To this end, we impose the following assumption.

Assumption 4 (Convexity and constraint qualification)**.** The following conditions hold.

(i) $\mathcal{Y} = \mathbb{R}^q$.

(ii) For every $\mathbf{x}\in\mathcal{X}$ and $\boldsymbol{\xi}\in\Xi$, the functions $\mathbb{E}_{\boldsymbol{\zeta}}[Q(\mathbf{x},\mathbf{y},\boldsymbol{\xi},\boldsymbol{\zeta})]$ and $H_j(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ for $j = 1,\ldots,r$ are convex and continuously differentiable.

(iii) For every $\mathbf{x}\in\mathcal{X}$ and $\boldsymbol{\xi}\in\Xi$, the follower problem admits an optimal solution and satisfies Slater's condition; that is, there exists $\bar{\mathbf{y}}\in\mathbb{R}^q$ such that $\mathbf{H}(\mathbf{x},\bar{\mathbf{y}},\boldsymbol{\xi}) < 0$.

Remark 5. The assumption $\mathcal{Y} = \mathbb{R}^q$ is made for simplicity of exposition. If $\mathcal{Y}$ can be represented by finitely many convex and continuously differentiable inequality constraints involving only $\mathbf{y}$, these constraints can be incorporated into $\mathbf{H}(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) \leq 0$. Consequently, we may take $\mathcal{Y} = \mathbb{R}^q$ in the KKT-based reformulation of (16) without loss of generality, provided that the remaining assumptions hold for the augmented constraint system.

Under Assumption 4, the KKT conditions are necessary and sufficient for the optimality of the follower's response. Introducing the dual variables $\boldsymbol{\lambda}(\boldsymbol{\xi}) \in \mathbb{R}^r$ associated with the follower's constraints, these conditions can be written as

$$\boldsymbol{\lambda}(\boldsymbol{\xi})^\top \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0, \ \forall \boldsymbol{\xi} \in \Xi, \tag{22}$$

$$\nabla_{\mathbf{y}} [\mathbb{E}_{\boldsymbol{\zeta}} Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})] + [\nabla_{\mathbf{y}} \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})]^\top \boldsymbol{\lambda}(\boldsymbol{\xi}) = 0, \ \forall \boldsymbol{\xi} \in \Xi, \tag{23}$$

$$\boldsymbol{\lambda}(\boldsymbol{\xi}) \geq 0, \quad \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq 0, \ \forall \boldsymbol{\xi} \in \Xi. \tag{24}$$

To obtain an exact Big-M reformulation of the complementary slackness conditions, we further impose the following boundedness assumption.

Assumption 6 (Uniform Big-M bounds)**.** There exists a constant $M > 0$ such that, for every $\mathbf{x} \in \mathcal{X}$, $\boldsymbol{\xi} \in \Xi$, and every $\mathbf{y}(\boldsymbol{\xi})$ satisfying the KKT conditions (22)–(24) for some multiplier, there exists an associated KKT multiplier $\boldsymbol{\lambda}(\boldsymbol{\xi})$ satisfying $0 \leq \boldsymbol{\lambda}(\boldsymbol{\xi}) \leq M\mathbf{1}$ and $-M\mathbf{1} \leq \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq 0$.

Under Assumption 6, the complementary slackness constraints (22) can be reformulated by introducing auxiliary binary variables as follows:

$$\boldsymbol{\lambda}(\boldsymbol{\xi}) \leq M\mathbf{w}(\boldsymbol{\xi}), \ M(\mathbf{1} - \mathbf{w}(\boldsymbol{\xi})) \leq \mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}), \ \mathbf{w}(\boldsymbol{\xi}) \in \{0,1\}^r, \ \forall \boldsymbol{\xi} \in \Xi. \tag{25}$$

Combining the above reformulations, we obtain the following result.

Theorem 7 (Exact MINLP reformulation)**.** *Suppose Assumptions 4 and 6 hold. Then problem (16), and hence the DFBO model (3), can be equivalently reformulated as the following mixed-integer nonlinear programming (MINLP) problem:*

$$\min_{\substack{(\mathbf{x}, \mathbf{y}(\cdot), \nu) \in \mathcal{F}' \\ \boldsymbol{\lambda}(\cdot), \mathbf{w}(\cdot)}} \ \nu \quad s.t. \quad \mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\zeta}}[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})] \leq c, \ (23), (24), (25), \tag{26}$$

where $\mathcal{F}'$ is the equivalent quantile-based representation of $\mathcal{F}$ established in Lemma 3.

Proof. See Section A.2 in the Appendix. □

Remark 8. Suppose Assumption 4 holds except that $\mathbb{E}_{\boldsymbol{\zeta}}[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})]$ or the components of $\mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ are nonsmooth with respect to $\mathbf{y}$. The stationarity condition (23) can be replaced by

$$\mathbf{0} \in \partial_{\mathbf{y}} \mathbb{E}_{\boldsymbol{\zeta}}[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})] + \sum_{j=1}^{r} \lambda_j(\boldsymbol{\xi}) \partial_{\mathbf{y}} H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}), \qquad \forall \boldsymbol{\xi} \in \Xi, \tag{27}$$

where $\partial_{\mathbf{y}}$ denotes the subdifferential with respect to $\mathbf{y}$ and $\boldsymbol{\lambda}(\boldsymbol{\xi}) = (\lambda_1(\boldsymbol{\xi}), \ldots, \lambda_r(\boldsymbol{\xi}))^\top$. Under additional structural conditions, the inclusion in (27) still admits a mixed-integer representation, so (3) can still be reformulated as an MINLP problem. For example, if $\mathbb{E}_{\boldsymbol{\zeta}}[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})]$ and the components of $\mathbf{H}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ are convex piecewise affine functions of $\mathbf{x}$ and $\mathbf{y}(\boldsymbol{\xi})$ with finitely many pieces, (27) can be represented equivalently by linear equalities and inequalities involving auxiliary continuous and binary variables, provided that valid bounds are available for the required big-M reformulations.

Additionally, we note that problem (26) is a MINLP, but under certain conditions this problem can be represented as a mixed-integer convex program.

Proposition 9. *Suppose Assumptions 4 and 6 hold, and $\mathcal{X}$ is a bounded polyhedron. For every fixed realization of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, suppose that G and U_s, $s \in \mathcal{S}$, are continuous piecewise affine functions of $(\mathbf{x}, \mathbf{y})$ with finitely many pieces, and that Q and each component of $\mathbf{H}$ are convex piecewise affine functions of $(\mathbf{x}, \mathbf{y})$ with finitely many pieces. Then model (26) admits an equivalent mixed-integer convex quadratically constrained formulation.*

Proof. See Section A.3 in the Appendix. □

3.2 Relaxation of DFBO

Solving the exact MINLP formulation (26) to global optimality can be computationally prohibitive, particularly for large-scale instances. In practice, one may therefore rely on more scalable solution methods, such as the first-order approach developed in the next section, to obtain high-quality solutions without guarantees of global optimality. To assess the quality of feasible solutions obtained by such methods and quantify their potential suboptimality, it is important to establish a valid lower bound on the optimal objective value. To this end, we first derive a lower bound on the Wasserstein unfairness measure $\mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot))$ using Jensen's inequality motivated by Ye et al. (2024).

Lemma 10 (Jensen Bound)**.** *For any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y}(\cdot) \in \mathcal{Y}^{N_{\boldsymbol{\xi}}}$, it holds that*

$$\mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot)) \geq \max_{s,s' \in \mathcal{S}} \left\{\mathbb{E}[U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})] - \mathbb{E}[U_{s'}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})\right\}^2. \tag{28}$$

Proof. Recall Definition 1. Fix any $s, s' \in \mathcal{S}$ and let $\mathcal{Q}_{\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}}$ be the set of joint distributions of U_s and $U_{s'}$ with marginals $\mathbb{P}_{U_s}$ and $\mathbb{P}_{U_{s'}}$. For any $\mathbb{Q} \in \mathcal{Q}_{\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}}$, by Jensen's inequality,

$$\mathbb{E}_{\mathbb{Q}}\left[|U_s - U_{s'}|^2\right] \geq |\mathbb{E}_{\mathbb{Q}}[U_s - U_{s'}]|^2 = \left|\mathbb{E}_{\mathbb{P}_{U_s}}[U_s] - \mathbb{E}_{\mathbb{P}_{U_{s'}}}[U_{s'}]\right|^2 = |\mathbb{E}[U_s] - \mathbb{E}[U_{s'}]|^2.$$

Taking the infimum over $\mathbb{Q} \in \mathcal{Q}_{\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}}$ yields $\mathbb{W}_2^2(\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}) \geq |\mathbb{E}[U_s] - \mathbb{E}[U_{s'}]|^2$. Taking the maximum over all $s, s' \in \mathcal{S}$ establishes (28). □

Using Lemma 10, we obtain the following lower-bounding formulation of DFBO.

Theorem 11 (Jensen-Based Lower Bound for DFBO)**.** *Consider the following problem:*

$$\begin{aligned} \min_{\mathbf{x} \in \mathcal{X},\ \mathbf{y}(\cdot)} \quad & \max_{s,s' \in \mathcal{S}} \left\{\mathbb{E}[U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})] - \mathbb{E}[U_{s'}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})\right\}^2 \\ s.t. \quad & \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})\right] \leq c, \\ & \mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}' \in \mathcal{Y}} \left\{\mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}, \boldsymbol{\zeta})\right] : \mathbf{H}(\mathbf{x}, \mathbf{y}', \boldsymbol{\xi}) \leq 0\right\}, \ \forall \boldsymbol{\xi} \in \Xi. \end{aligned} \tag{29}$$

Let v_{J}^* denote the optimal objective value of (29) and v_{DFBO}^* denotes the optimal objective value of (3). Then $v_{\mathrm{J}}^* \leq v_{\mathrm{DFBO}}^*$.

Proof. Since (29) and (3) have the same feasible set and, by Lemma 10, $\max_{s,s'} |\mathbb{E}[U_s] - \mathbb{E}[U_{s'}]|^2 \leq \mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot))$ for any feasible solution, the conclusion follows. □

Remark 12. Although (29) provides a valid lower bound on the optimal objective value of DFBO, it is not necessarily computationally tractable due to the potentially nonconvex constraints, which further depends on the structure of $U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})$. Nevertheless, for specific applications, the objective function of (29) is convex, such that additional problem-specific relaxations of the feasible set of (29) may yield a tractable convex optimization model whose optimal value is the lower bound

of v^*_{DFBO}. In particular, we develop such a relaxation along with a relaxation of the lower-level optimality conditions for the capacity-planning problem in Appendix E and use the resulting computable lower bound to evaluate solution quality in Section 6.

4 DC Programming and Nearly Stationary Point

Although the MINLP reformulation (26) enables global optimization in principle, its computational cost can become prohibitive for large instances. We therefore pursue a scalable first-order algorithm for (3) to obtain high-quality solutions without requiring global optimality. However, direct application of first-order methods is challenging because the objective and constraints, particularly the lower-level optimality constraints, may be nonsmooth and nonconvex. To address these challenges, we impose structural assumptions under which (3) admits an equivalent difference-of-convex (DC) reformulation. This structure enables us to develop an inexact DC algorithm using first-order information, with provable guarantees for approximate stationarity.

4.1 DC Reformulation of the Objective of (3)

We first show that the objective of (3) admits a DC representation under mild structural assumptions on the utility functions U_s's. We impose the following assumption.

Assumption 13 (DC-representability of pairwise squared utility gaps)**.** For every $s, s' \in \mathcal{S}$ and every $\ell, \ell' \in \mathcal{L}$, the squared utility gap admits a DC decomposition

$$\left[u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - u_{s',\ell'}(\mathbf{x}, \mathbf{y}(\cdot))\right]^2 = d^+_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)) - d^-_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)),$$

where $d^+_{ss',\ell\ell'}$ and $d^-_{ss',\ell\ell'}$ are convex functions of $(\mathbf{x}, \mathbf{y}(\cdot))$.

Proposition 14 (DC representation of the objective of (3))**.** *Suppose Assumption 13 holds. It holds that*

$$\mathbb{U}(\mathbf{x}, \mathbf{y}(\cdot)) = \mathbb{U}^+(\mathbf{x}, \mathbf{y}(\cdot)) - \mathbb{U}^-(\mathbf{x}, \mathbf{y}(\cdot)),$$

where $\mathbb{U}^+$ and $\mathbb{U}^-$ are convex functions of $(\mathbf{x}, \mathbf{y}(\cdot))$. Moreover, their function values and subgradients are computable whenever those of $d^+_{ss',\ell\ell'}$ and $d^-_{ss',\ell\ell'}$ in Assumption 13 are computable.

Proof. See Section A.4 in the Appendix. □

Remark 15 (Verification of Assumption 13 for the applications). Assumption 13 holds for both applications in Section 2. We verify this claim below.

First, consider the capacity planning problem in Section 2.2. Since there is no $\boldsymbol{\zeta}$, $\mathcal{L}$ is essentially reduced to $\{1, \ldots, N_{\boldsymbol{\xi}}\}$. By (8) and (14), for each $\ell \in \mathcal{L}$, $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) = \frac{1}{m_s} \sum_{j=1}^{m_s} \left(\xi^s_{\ell,j} - \sum_{i=1}^n y^s_{ij}(\boldsymbol{\xi}_\ell)\right)$, where $\xi^s_{\ell,j}$ denotes the demand of customer j in community s under scenario $\boldsymbol{\xi}_\ell$. Hence, for any $s, s' \in \mathcal{S}$ and any $\ell, \ell' \in \mathcal{L}$, the squared utility gap $\left[u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - u_{s',\ell'}(\mathbf{x}, \mathbf{y}(\cdot))\right]^2$ is a convex quadratic function of $(\mathbf{x}, \mathbf{y}(\cdot))$. Thus, Assumption 13 holds with the simple decomposition $d^+_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)) = \left[u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - u_{s',\ell'}(\mathbf{x}, \mathbf{y}(\cdot))\right]^2$, $d^-_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)) = 0$.

Second, consider the newsvendor pricing game in Section 2.3. Since there is no $\boldsymbol{\xi}$, $\mathcal{L}$ is essentially reduced to $\{1, \ldots, N_{\boldsymbol{\zeta}}\}$. By (12) and (14), for each $\ell \in \mathcal{L}$, $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) = (\zeta_{\ell,s} - y_s)_+$, where $\zeta_{\ell,s}$ denotes the demand of community s under scenario $\boldsymbol{\zeta}_\ell$. Using the identity $(a-b)^2 = 2a^2 + 2b^2 - (a+b)^2$, for any $s, s' \in \mathcal{S}$ and any $\ell, \ell' \in \mathcal{L}$, the squared utility gap can be represented as

$$\left[u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - u_{s',\ell'}(\mathbf{x}, \mathbf{y}(\cdot))\right]^2 = d^+_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)) - d^-_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}(\cdot)),$$

where $d^+_{ss',\ell\ell'}(\mathbf{x},\mathbf{y}(\cdot)) := 2(\zeta_{\ell,s}-y_s)^2_+ + 2(\zeta_{\ell',s'}-y_{s'})^2_+$ and $d^-_{ss',\ell\ell'}(\mathbf{x},\mathbf{y}(\cdot)) := \left((\zeta_{\ell,s}-y_s)_+ + (\zeta_{\ell',s'}-y_{s'})_+\right)^2$. Here, both $d^+_{ss',\ell\ell'}$ and $d^-_{ss',\ell\ell'}$ are convex in $(\mathbf{x},\mathbf{y}(\cdot))$, and Assumption 13 holds.

4.2 DC Reformulation of the Constraints of (3)

We next introduce structural assumptions under which (3) admits a reformulation as an optimization problem with DC constraints.

Assumption 16 (DC-representability of upper-level cost)**.** The upper-level cost admits a DC decomposition $C(\mathbf{x},\mathbf{y}(\cdot)) := \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi},\boldsymbol{\zeta})\right] = C^+(\mathbf{x},\mathbf{y}(\cdot)) - C^-(\mathbf{x},\mathbf{y}(\cdot))$ where C^+ and C^- are convex functions of $(\mathbf{x},\mathbf{y}(\cdot))$

Assumption 17 (DC-representability of lower-level feasibility)**.** For each $\boldsymbol{\xi} \in \Xi$ and each $k = 1,\ldots,r$, the constraint function $H_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ admits a DC decomposition $H_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = H^+_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) - H^-_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$, where $H^+_k(\cdot,\cdot,\boldsymbol{\xi})$ and $H^-_k(\cdot,\cdot,\boldsymbol{\xi})$ are convex.

Next, before we present the assumption under which the lower-level optimality condition

$$\mathbf{y}(\boldsymbol{\xi}) \in \arg\min_{\mathbf{y}'\in\mathcal{Y}} \left\{\mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x},\mathbf{y}',\boldsymbol{\xi},\boldsymbol{\zeta})\right] : \mathbf{H}(\mathbf{x},\mathbf{y}',\boldsymbol{\xi}) \le 0\right\} \tag{30}$$

in (3) can be formulated as a DC constraint, we first define the lower-level objective $q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) := \mathbb{E}_{\boldsymbol{\zeta}}\left[Q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi},\boldsymbol{\zeta})\right]$, and the lower-level value function

$$\phi(\mathbf{x},\boldsymbol{\xi}) := \min_{\mathbf{y}\in\mathcal{Y}} \left\{q(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) : \mathbf{H}(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) \le 0\right\}$$

for each $\boldsymbol{\xi} \in \Xi$. Then constraint (30) can be equivalently written as

$$\mathbf{y}(\boldsymbol{\xi}) \in \mathcal{Y}, \quad \mathbf{H}(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) \le 0, \quad q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) - \phi(\mathbf{x},\boldsymbol{\xi}) \le 0, \qquad \forall \boldsymbol{\xi} \in \Xi. \tag{31}$$

Indeed, by definition of $\phi(\mathbf{x},\boldsymbol{\xi})$, every feasible $\mathbf{y}(\boldsymbol{\xi})$ satisfies $q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) \ge \phi(\mathbf{x},\boldsymbol{\xi})$; hence the above inequality holds if and only if $\mathbf{y}(\boldsymbol{\xi})$ attains the lower-level optimal value. This is the standard value-function reformulation of the lower-level optimality condition.

Assumption 18 (DC-representability of lower-level optimality)**.** For each $\boldsymbol{\xi} \in \Xi$, the lower-level objective $q(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ admits a DC decomposition $q(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = q^+(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) - q^-(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$, where $q^+(\cdot,\cdot,\boldsymbol{\xi})$ and $q^-(\cdot,\cdot,\boldsymbol{\xi})$ are convex. Moreover, the lower-level value function $\phi(\mathbf{x},\boldsymbol{\xi})$ admits a DC decomposition $\phi(\mathbf{x},\boldsymbol{\xi}) = \phi^+(\mathbf{x},\boldsymbol{\xi}) - \phi^-(\mathbf{x},\boldsymbol{\xi})$, where $\phi^+(\cdot,\boldsymbol{\xi})$ and $\phi^-(\cdot,\boldsymbol{\xi})$ are convex.

Theorem 19 (DC reformulation of (3))**.** *Suppose Assumptions 16–18 hold. Then* (3) *becomes*

$$\begin{aligned}
\min_{\mathbf{x}\in\mathcal{X},\mathbf{y}(\cdot)\in\mathcal{Y}^{N_\Xi}} \quad & \mathbb{U}(\mathbf{x},\mathbf{y}(\cdot)) \\
\text{s.t.} \quad & C^+(\mathbf{x},\mathbf{y}(\cdot)) - C^-(\mathbf{x},\mathbf{y}(\cdot)) \le c, \\
& H^+_k(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) - H^-_k(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) \le 0, \quad \forall \boldsymbol{\xi} \in \Xi, \quad \textit{for } k = 1,\ldots,r, \\
& q^+(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) + \phi^-(\mathbf{x},\boldsymbol{\xi}) - \left[q^-(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi}) + \phi^+(\mathbf{x},\boldsymbol{\xi})\right] \le 0, \quad \forall \boldsymbol{\xi} \in \Xi.
\end{aligned} \tag{32}$$

In particular, the objective and constraint functions in (32) *are DC.*

Proof. For each $\boldsymbol{\xi} \in \Xi$, the lower-level optimality condition (30) is equivalent to (31) which are all DC constraints according to Assumption 17 and Assumption 18. Moreover, the upper-level cost constraint is DC by Assumption 16 and the objective $\mathbb{U}(\mathbf{x},\mathbf{y}(\cdot))$ is DC by Proposition 14. □

Remark 20 (Verification of Assumptions 16–18 for the applications). The above framework covers both applications in Section 2.

In the distributionally fair capacity planning problem in Section 2.2, the upper-level cost $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi},\boldsymbol{\zeta})\right]$ and the lower-level constraints $\mathbf{H}(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi})$ are all linear in $(\mathbf{x},\mathbf{y}(\cdot))$, so Assumptions 16 and 17 hold. Additionally, the lower-level objective $q(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ is a linear function of only $\mathbf{y}$, so it is easy to show that the value function $\phi(\mathbf{x},\boldsymbol{\xi})$ is convex in $\mathbf{x}$ for any $\boldsymbol{\xi}\in\Xi$, and thus Assumption 18 holds. See also Section B.

In the newsvendor pricing game in Section 2.3, there are no lower-level inequality constraints $\mathbf{H}$ in the lower-level problem, so Assumption 17 holds trivially. Since there is no $\boldsymbol{\xi}$, we write the lower-level objective and the lower-level value function as $q(\mathbf{x},\mathbf{y})$ and $\phi(\mathbf{x})$, respectively. Since the only non-convex term in $q(\mathbf{x},\mathbf{y})$ and in the upper-level cost $C(\mathbf{x},\mathbf{y})$ is $\mathbf{x}^\top\mathbf{y}$, they both admits DC decompositions $C(\mathbf{x},\mathbf{y})=C^+(\mathbf{x},\mathbf{y})-C^-(\mathbf{x},\mathbf{y})$, $q(\mathbf{x},\mathbf{y})=q^+(\mathbf{x},\mathbf{y})-q^-(\mathbf{x},\mathbf{y})$ where $C^+(\mathbf{x},\mathbf{y})=(\mathbf{c}-\mathbf{x})^\top\mathbf{y}+\frac{1}{2}\|\mathbf{x}\|^2+\frac{1}{2}\|\mathbf{y}\|^2$ and $C^-(\mathbf{x},\mathbf{y})=\frac{1}{2}\|\mathbf{x}\|^2+\frac{1}{2}\|\mathbf{y}\|^2$ are both convex, and $q^+(\mathbf{x},\mathbf{y})=q(\mathbf{x},\mathbf{y})+\frac{1}{2}\|\mathbf{x}\|^2+\frac{1}{2}\|\mathbf{y}\|^2$ and $C^-(\mathbf{x},\mathbf{y})=\frac{1}{2}\|\mathbf{x}\|^2+\frac{1}{2}\|\mathbf{y}\|^2$ are also convex. Since $q(\mathbf{x},\mathbf{y})$ is affine in $\mathbf{x}$ for any given $\mathbf{y}$, $\phi(\mathbf{x})$ is concave in $\mathbf{x}$. This means Assumptions 16 and 18 holds.

To apply first-order methods to the DC reformulation (32), one needs to evaluate the value functions $\phi^+(\mathbf{x},\boldsymbol{\xi})$ and $\phi^-(\mathbf{x},\boldsymbol{\xi})$, as well as their subgradients with respect to $\mathbf{x}$. In general, this task can be challenging because these value functions are defined implicitly through lower-level optimization problems. However, in important applications with additional structure, the value functions and their subgradients can be computed exactly in polynomial time. In Section B in Appendix, we illustrate this point using the two applications from Section 2.

4.3 Nearly ϵ-KKT Points

Suppose Assumptions 13–18 hold. By Propositions 14 and 19, the sample average approximation of the distributionally fair bilevel optimization problem in (3) can be reformulated as the DC-constrained problem in (32). Because this reformulated problem is nonconvex, our goal is to compute an approximate stationary point rather than a global optimizer.

For the convergence analysis, we also allow a positive relaxation $\hat{\epsilon}>0$ of the lower-level optimality constraints, replacing $q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi})-\phi(\mathbf{x},\boldsymbol{\xi})\le 0$ by $q(\mathbf{x},\mathbf{y}(\boldsymbol{\xi}),\boldsymbol{\xi})-\phi(\mathbf{x},\boldsymbol{\xi})\le\hat{\epsilon}$. The resulting problem can still be written in the following generic DC form.

To simplify notation, let $\mathbf{w}=(\mathbf{x},\mathbf{y}(\cdot))$ and write the DC reformulation generically as

$$\min_{\mathbf{w}\in\mathcal{W}}\ f_0(\mathbf{w}):=f_0^+(\mathbf{w})-f_0^-(\mathbf{w})\ \text{ s.t. }\ f_i(\mathbf{w}):=f_i^+(\mathbf{w})-f_i^-(\mathbf{w})\le 0,\quad i\in[m], \tag{33}$$

where $\mathcal{W}$ is a convex closed set, and f_i^+ and f_i^-, $i=0,1,\ldots,m$, are finite-valued convex functions, possibly nonsmooth. Let $\mathcal{N}_{\mathcal{W}}(\mathbf{w})$ be the normal cone of $\mathcal{W}$ at $\mathbf{w}\in\mathcal{W}$. We use the following approximate stationarity notions.

Definition 21 (ϵ-KKT point)**.** A point $\mathbf{w}\in\mathcal{W}$ is an *ϵ-KKT point* of (33) if there exist $\mathbf{n}_{\mathcal{W}}\in\mathcal{N}_{\mathcal{W}}(\mathbf{w})$, $\mathbf{f}_i^+\in\partial f_i^+(\mathbf{w})$, $\mathbf{f}_i^-\in\partial f_i^-(\mathbf{w})$, $e_i\in\mathbb{R}$, and $\lambda_i\ge 0$ for $i\in[m]$ such that

$$\left\|\mathbf{f}_0^+-\mathbf{f}_0^-+\sum_{i=1}^m\lambda_i(\mathbf{f}_i^+-\mathbf{f}_i^-)+\mathbf{n}_{\mathcal{W}}\right\|\le 0, \tag{34}$$

$$f_i(\mathbf{w})\le e_i,\quad i\in[m], \tag{35}$$

$$\lambda_i(f_i(\mathbf{w})-e_i)=0,\quad i\in[m], \tag{36}$$

$$|e_i|\le\epsilon,\quad i\in[m]. \tag{37}$$

Algorithm 1 Inexact DC Algorithm (IDCA) for (33)

Input: A feasible point $\mathbf{w}^{(0)}$, number of outer iterations K, precisions $\{\epsilon_k\}_{k=0}^{K-1}$, and perturbations $\{\delta_k\}_{k=0}^{K-1}$.
for $k = 0, 1, \ldots, K-1$ **do**
    Find $\mathbf{w}^{(k+1)}$ satisfying (41) by approximately solving (38).
end for

Definition 21 can be interpreted as an exact KKT condition for a perturbed problem with constraints $f_i(\mathbf{w}) \le e_i$, where $|e_i| \le \epsilon$ for all $i \in [m]$.

Definition 22 (Nearly ϵ-KKT point)**.** A point $\mathbf{w} \in \mathcal{W}$ is a *nearly ϵ-KKT point* of (33) if there exists $\widehat{\mathbf{w}} \in \mathcal{W}$ such that $\|\widehat{\mathbf{w}} - \mathbf{w}\| \le \epsilon$ and there exist $\mathbf{n}_{\mathcal{W}} \in \mathcal{N}_{\mathcal{W}}(\widehat{\mathbf{w}})$, $\mathbf{f}_i^+ \in \partial f_i^+(\widehat{\mathbf{w}})$, $\mathbf{f}_i^- \in \partial f_i^-(\mathbf{w})$, $e_i \in \mathbb{R}$, and $\lambda_i \ge 0$ for $i \in [m]$ such that (34)–(37) hold.

Definition 22 differs from Definition 21 in that the subgradients of f_i^+ and the normal vector are evaluated at the nearby point $\widehat{\mathbf{w}}$, whereas the subgradients of f_i^- and the values of f_i are evaluated at $\mathbf{w}$.

5 Inexact DC Algorithm and its Complexity Analysis

We apply an *inexact DC algorithm* (IDCA) to compute a nearly ϵ-KKT point of (33). Given the current iterate $\mathbf{w}^{(k)} \in \mathcal{W}$, we linearize the concave component of each DC function at $\mathbf{w}^{(k)}$ and solve the following convex approximation:

$$\begin{aligned} \min_{\mathbf{v} \in \mathcal{W}} \; & g_0(\mathbf{v}; \mathbf{w}^{(k)}) := f_0^+(\mathbf{v}) - \widehat{f}_0^-(\mathbf{v}; \mathbf{w}^{(k)}) \\ \text{s.t.} \; & g_i(\mathbf{v}; \mathbf{w}^{(k)}) := f_i^+(\mathbf{v}) - \widehat{f}_i^-(\mathbf{v}; \mathbf{w}^{(k)}) \le \delta_k, \quad i \in [m], \end{aligned} \tag{38}$$

where $\delta_k \in \mathbb{R}$ is a *perturbation parameter* and

$$\widehat{f}_i^-(\mathbf{v}; \mathbf{w}) := f_i^-(\mathbf{w}) + \langle \mathbf{f}_i^-, \mathbf{v} - \mathbf{w} \rangle, \qquad \mathbf{f}_i^- \in \partial f_i^-(\mathbf{w}). \tag{39}$$

By convexity of f_i^-, we have

$$f_i(\mathbf{v}) \le g_i(\mathbf{v}; \mathbf{w}^{(k)}), \qquad i = 0, 1, \ldots, m. \tag{40}$$

Let $\widehat{\mathbf{w}}^{(k+1)}$ denote an optimal solution of (38). Instead of solving (38) exactly, IDCA computes an approximate solution $\mathbf{w}^{(k+1)}$ satisfying

$$\max \left\{ g_0(\mathbf{w}^{(k+1)}; \mathbf{w}^{(k)}) - g_0(\widehat{\mathbf{w}}^{(k+1)}; \mathbf{w}^{(k)}),\; g(\mathbf{w}^{(k+1)}; \mathbf{w}^{(k)}) - \delta_k \right\} \le \epsilon_k, \tag{41}$$

where

$$g(\mathbf{v}; \mathbf{w}^{(k)}) := \max_{i \in [m]} g_i(\mathbf{v}; \mathbf{w}^{(k)}).$$

The resulting method is summarized in Algorithm 1.

We impose the following assumptions on (33) for the convergence analysis.

Assumption 23. The following conditions hold.

A. The functions f_0^+ and f_0^- are finite-valued and μ-strongly convex on $\mathbb{R}^n$ with $\mu > 0$.

B. There exists $\bar{\delta} > 0$ such that, for any $\delta \in (0, \bar{\delta}]$, there exist constants $\epsilon_\delta > 0$ and $\nu_\delta > 0$ such that, for every $\mathbf{w} \in \mathcal{W}$ satisfying

$$f_i(\mathbf{w}) - \delta \le \epsilon_\delta^2, \qquad i \in [m],$$

there exists $\mathbf{v} \in \mathrm{int}(\mathcal{W})$ satisfying

$$g_i(\mathbf{v}; \mathbf{w}) - \delta \le -\nu_\delta, \qquad i \in [m].$$

C. The set $\mathcal{W}$ is convex and compact. Moreover,

$$D := \max_{\mathbf{w}, \mathbf{w}' \in \mathcal{W}} \|\mathbf{w} - \mathbf{w}'\| < +\infty \quad \text{and} \quad f_{\mathrm{lb}} := \min_{\mathbf{w} \in \mathcal{W}} f_0(\mathbf{w}) > -\infty.$$

D. For each $i = 0, 1, \ldots, m$, the functions f_i^+ and f_i^- are M-Lipschitz continuous on $\mathcal{W}$.

E. An initial feasible solution $\mathbf{w}_{\mathrm{feas}} \in \mathcal{W}$ satisfying $f_i(\mathbf{w}_{\mathrm{feas}}) \le 0$ for all $i \in [m]$ is available.

Assumption 23 is satisfied with $\epsilon_\delta = \Theta(\delta)$ and $\nu_\delta = \Theta(\delta)$ by the empirical DC formulations of both applications in Section 2 under additional mild conditions. The detailed verification is provided in Appendix C.

5.1 Outer Iteration Complexity

For later use, we define the convex subproblem generated at a generic point $\mathbf{w} \in \mathcal{W}$:

$$\begin{aligned} \min_{\mathbf{v} \in \mathcal{W}} \; & g_0(\mathbf{v}; \mathbf{w}) := f_0^+(\mathbf{v}) - \widehat{f}_0^-(\mathbf{v}; \mathbf{w}) \\ \text{s.t.} \; & g_i(\mathbf{v}; \mathbf{w}) := f_i^+(\mathbf{v}) - \widehat{f}_i^-(\mathbf{v}; \mathbf{w}) \le \delta, \quad i \in [m], \end{aligned} \tag{42}$$

where $\delta \ge 0$ and $\widehat{f}_i^-$ is defined in (39). By Assumption 23B, if $\delta \in (0, \bar{\delta}]$ and $f_i(\mathbf{w}) - \delta \le \epsilon_\delta^2$ for all $i \in [m]$, Slater's condition holds for (42) with margin ν_δ. In this case, let $\widehat{\mathbf{w}}$ be an optimal solution of (42). Then there exists a vector of Lagrange multipliers $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)^\top \in \mathbb{R}_+^m$ such that

$$\mathbf{f}_0^+ - \mathbf{f}_0^- + \sum_{i=1}^m \lambda_i (\mathbf{f}_i^+ - \mathbf{f}_i^-) + \mathbf{n}_{\mathcal{W}} = \mathbf{0}, \tag{43}$$

$$g_i(\widehat{\mathbf{w}}; \mathbf{w}) - \delta_\epsilon \le 0, \quad i \in [m], \tag{44}$$

$$\lambda_i (g_i(\widehat{\mathbf{w}}; \mathbf{w}) - \delta_\epsilon) = 0, \quad i \in [m], \tag{45}$$

where

$$\mathbf{n}_{\mathcal{W}} \in \mathcal{N}_{\mathcal{W}}(\widehat{\mathbf{w}}), \quad \mathbf{f}_i^+ \in \partial f_i^+(\widehat{\mathbf{w}}), \quad \mathbf{f}_i^- \in \partial f_i^-(\mathbf{w}), \qquad i = 0, 1, \ldots, m.$$

The next result shows that if the exact solution of the convexified subproblem (42) is sufficiently close to the current iterate, then the current iterate is nearly stationary for the original DC program.

Lemma 24. *Suppose* $0 < \delta \le \min\{\bar{\delta}, \epsilon/2\}$ *and* $f_i(\mathbf{w}) - \delta \le \epsilon_\delta^2$ *for all* $i \in [m]$, *and*

$$\|\widehat{\mathbf{w}} - \mathbf{w}\| \le \min\left\{1, \frac{1}{4M}\right\} \epsilon, \tag{46}$$

where $\widehat{\mathbf{w}}$ *is an optimal solution of* (42). *Then* $\mathbf{w}$ *is a nearly* ϵ*-KKT point of* (33).

Proof. See Section D.1 in the Appendix. □

The following lemma provides a uniform bound on the Lagrange multipliers of (42). This bound is essential for controlling the complexity of finding a solution of (38) that satisfies (41).

Lemma 25. *Suppose $0 < \delta \le \bar{\delta}$ and $f_i(\mathbf{w}) - \delta \le \epsilon_\delta^2$ for all $i \in [m]$. Let $\widehat{\mathbf{w}}$ be an optimal solution of (42), and let λ_i, $i \in [m]$, be the corresponding Lagrange multipliers. Then*

$$\|\widehat{\mathbf{w}} - \mathbf{w}\| \le \frac{2(1 + \sum_{i=1}^m \lambda_i)M}{\mu}, \qquad \sum_{i=1}^m \lambda_i \le \frac{2MD}{\nu_\delta}. \tag{47}$$

Proof. See Section D.2 in the Appendix. □

We are ready to present the main theorem on the convergence of DCA.

Theorem 26. *Suppose $0 < \delta_k \le \min\{\bar{\delta}, \epsilon/2\}$ and $\delta_k + \epsilon_k \le \delta_{k+1} + \epsilon_{\delta_{k+1}}^2$ for all $k \ge 0$. Then*

$$\min_{0 \le k' \le K-1} \left[\|\widehat{\mathbf{w}}^{(k'+1)} - \mathbf{w}^{(k')}\|^2\right] \le \frac{2(f_0(\mathbf{w}^{(0)}) - f_{\rm lb})}{K\mu} + \frac{2}{K\mu}\sum_{k=0}^{K-1}\epsilon_k + \frac{4MD}{K\mu}\sum_{k=0}^{K-1}\frac{(\delta_{k-1} + \epsilon_{k-1} - \delta_k)_+}{\nu_{\delta_k}}, \tag{48}$$

where $\delta_{-1} = \epsilon_{-1} = 0$. In particular, if $\delta_k = \delta_\epsilon := \min\{\bar{\delta}, \epsilon/2\}$,

$$K = \left\lceil \frac{4\max\{1, 16M^2\}\left(f_0(\mathbf{w}^{(0)}) - f_{\rm lb}\right)}{\mu\epsilon^2} \right\rceil, \tag{49}$$

$$\epsilon_k = \min\left\{\min\left\{1, \frac{1}{16M^2}\right\}\min\left\{1, \frac{\nu_{\delta_\epsilon}}{2MD}\right\}\frac{\mu\epsilon^2}{8}, \epsilon_{\delta_\epsilon}^2\right\}, \tag{50}$$

for any $k \ge 0$, then Algorithm 1 finds a nearly ϵ-KKT point of (33) within $K = O(\epsilon^{-2})$ outer iterations.

Proof. See Section D.3 in the Appendix. □

5.2 Inner Iteration Complexity

We use the ConEx method of Boob et al. (2023) to solve each convexified subproblem (38) because it has the optimal iteration complexity for this class of problem. For notational simplicity, for a fixed an outer iteration k, we write

$$g_i(\mathbf{v}) := g_i(\mathbf{v}; \mathbf{w}^{(k)}), \qquad \mathbf{g}_i(\mathbf{v}) \in \partial g_i(\mathbf{v}), \qquad i = 0, 1, \ldots, m.$$

and define the linearized constraint vector

$$\ell(\mathbf{v}_t) := \Big(g_1(\mathbf{v}_{t-1}) - \delta_k + \langle \mathbf{g}_1(\mathbf{v}_{t-1}), \mathbf{v}_t - \mathbf{v}_{t-1}\rangle, \ldots, g_m(\mathbf{v}_{t-1}) - \delta_k + \langle \mathbf{g}_m(\mathbf{v}_{t-1}), \mathbf{v}_t - \mathbf{v}_{t-1}\rangle\Big)^\top. \tag{51}$$

The ConEx method is stated in Algorithm 2.

The following result follows from Theorem 1 of Boob et al. (2023).

Algorithm 2 ConEx Method for Subproblem (38)

Input: $(\mathbf{v}_0, \mathbf{u}_0)$, parameters $\{(\gamma_t, \tau_t, \eta_t, \theta_t)\}_{t\geq 0}$, and number of inner iterations T_k.

1: $\mathbf{v}_{-1} \leftarrow \mathbf{v}_0$ and $\mathbf{v}_{-2} \leftarrow \mathbf{v}_0$.
2: **for** $t = 0, \ldots, T_k - 1$ **do**
3: $\quad \mathbf{s}_t \leftarrow (1+\theta_t)\ell(\mathbf{v}_t) - \theta_t \ell(\mathbf{v}_{t-1})$.
4: $\quad \mathbf{u}_{t+1} \leftarrow \arg\min_{\mathbf{u}\geq 0} \{\langle -\mathbf{s}_t, \mathbf{u}\rangle + \frac{\tau_t}{2}\|\mathbf{u} - \mathbf{u}_t\|_2^2\}$.
5: $\quad \mathbf{v}_{t+1} \leftarrow \arg\min_{\mathbf{v}\in\mathcal{W}} \left\{\left\langle \mathbf{g}_0(\mathbf{v}_t) + \sum_{i\in[m]} \mathbf{g}_i(\mathbf{v}_t) u_{i,t+1}, \mathbf{v}\right\rangle + \frac{\eta_t}{2}\|\mathbf{v} - \mathbf{v}_t\|_2^2\right\}$.
6: **end for**
7: $\mathbf{w}^{(k+1)} \leftarrow \left(\sum_{t=0}^{T_k-1} \gamma_t\right)^{-1} \sum_{t=0}^{T_k-1} \gamma_t \mathbf{v}_{t+1}$.

Theorem 27. *Let $\mathcal{M} := 4M\sqrt{m}$. Set $\mathbf{u}_0 = \mathbf{0}$, $T_k \geq 1$, and $(\gamma_t, \tau_t, \eta_t, \theta_t)$ in Algorithm 2 according to the following:*

$$\gamma_t = t+4,\ \tau_t = \frac{32\mathcal{M}^2}{\mu(t+1)},\ \eta_t = \frac{\mu(t+3)}{2},\ \theta_t = \frac{t+3}{t+4}.$$

Then Algorithm 2 guarantees

$$g_0(\mathbf{w}^{(k+1)}) - g_0(\widehat{\mathbf{w}}^{(k+1)}) \leq \frac{12\mu D^2}{T_k^2} + \frac{16(2e^2 H_*^2 + 16M^2)}{\mu T_k}, \tag{52}$$

and, for each $i \in [m]$,

$$g_i(\mathbf{w}^{(k+1)}) - \delta_k \leq \frac{768(\|\boldsymbol{\lambda}^{(k)}\|_2 + 1)^2\mathcal{M}^2}{\mu T_k^2} + \frac{12\mu D^2}{T_k^2} + \frac{16(2e^2+1)H_*^2}{\mu T_k}, \tag{53}$$

where $H_* := 2M + 2(\|\boldsymbol{\lambda}^{(k)}\|_2 + 1)M\sqrt{m}$ and $\widehat{\mathbf{w}}^{(k+1)}$ is an optimal solution of (38).

Proof. See Section D.4 in the Appendix. □

By Theorem 27, to obtain an ϵ_k-optimal and ϵ_k-feasible solution of (38), namely, to ensure (41), it suffices to take

$$T_k = O\left(\frac{H_*^2}{\epsilon_k}\right) = O\left(\frac{\|\boldsymbol{\lambda}^{(k)}\|_2^2}{\epsilon_k}\right)$$

iterations in Algorithm 2 for each outer iteration k.

5.3 Overall Complexity

Combining the outer and inner iteration complexity gives the overall complexity of DCA.

Theorem 28. *Suppose the assumptions of Theorems 26 and 27 hold. Suppose $\delta_k = \delta_\epsilon$ and (50) hold in Algorithm 1. Suppose also that ConEx in Algorithm 2 is applied to each subproblem (38) to ensure (41) by setting T_k large enough to ensure the left-hand sides of (52) and (53) are no more than ϵ_k. Then Algorithm 1 finds a nearly ϵ-KKT point of (33) within total inner-iteration complexity $O\left(\frac{1}{\nu_{\delta_\epsilon}^2 \epsilon^2 \min\{\nu_{\delta_\epsilon}\epsilon^2, \epsilon_{\delta_\epsilon}^2\}}\right)$.*

Proof. By Lemma 25, $\|\boldsymbol{\lambda}^{(k)}\|_2 \leq \sum_{i=1}^m \lambda_i^{(k)} \leq \frac{2MD}{\nu_{\delta_\epsilon}}$. Moreover, by (50), we have $\epsilon_k = \Theta(\min\{\nu_{\delta_\epsilon}\epsilon^2, \epsilon_{\delta_\epsilon}^2\})$. Therefore, Theorem 27 implies that the number of inner ConEx iterations required to ensure (41) at

each outer iteration satisfies $T_k = O\left(\frac{\|\boldsymbol{\lambda}^{(k)}\|_2^2}{\epsilon_k}\right) = O\left(\frac{1}{\nu_{\delta_\epsilon}^2 \min\{\nu_{\delta_\epsilon}\epsilon^2, \epsilon_{\delta_\epsilon}^2\}}\right)$. By Theorem 26, the number of outer IDCA iterations needed to find a nearly ϵ-KKT point of (33) is $K = O(\epsilon^{-2})$. Hence the total number of inner iterations is $\sum_{k=0}^{K-1} T_k = O\left(\frac{1}{\nu_{\delta_\epsilon}^2 \epsilon^2 \min\{\nu_{\delta_\epsilon}\epsilon^2, \epsilon_{\delta_\epsilon}^2\}}\right)$. □

Note that $\delta_\epsilon = \Theta(\epsilon)$. By Propositions 29 and 30, for the capacity planning problem (9) and the newsvendor pricing game (13), we have $\epsilon_\delta = \Theta(\delta)$ and $\nu_\delta = \Theta(\delta)$, so $\epsilon_{\delta_\epsilon} = \Theta(\epsilon)$ and $\nu_{\delta_\epsilon} = \Theta(\epsilon)$, which means the total iteration complexity of Algorithm 1 for finding a nearly ϵ-KKT point of (9) and (13) is $O\left(\frac{1}{\nu_{\delta_\epsilon}^2 \epsilon^2 \min\{\nu_{\delta_\epsilon}\epsilon^2, \epsilon_{\delta_\epsilon}^2\}}\right)) = O(\epsilon^{-7})$.

6 Numerical Results

In this section, we present the numerical performance of the proposed methods when applied to the capacity planning problem (9) and the newsvendor pricing game (13). The experiments are conducted on a computing node with 64 physical CPU cores, 512 GB of memory, using up to 32 threads. All experiments are implemented in Python 3.11 and solved using Gurobi 11.0.1. For both problems, we first describe the experimental instances and parameter settings, and then compare the solution quality and computational performance of the proposed methods with those of Gurobi.

6.1 Numerical Results for Capacity Planning Problem

6.1.1 Parameter Settings

We consider instances of (9) with the cost budget set to $c = \kappa c^*$ controlling distributional fairness level, where $\kappa \in \kappa^{CP} := \{1, 1.05, 1.10, 1.15, 1.20, 1.25, 1.30\}$ and c^* denotes the optimal objective value of the cost-only capacity-planning problem (7), which is a linear program and solved by Gurobi. We consider $S = 2$ communities and vary κ, $N_{\boldsymbol{\xi}}$, m_1, m_2, and n to generate instances of different sizes. For each instance, we set $g_i = 45$ and $o_i = 10$ for every production site $i = 1, \ldots, n$. The unit shipping costs c_{ij}^1 and c_{ij}^2 are sampled independently from the uniform distribution Unif$[5, 5.5]$, while the per-unit penalties for unmet demand are set to $p_1 = 6$ and $p_2 = 15$. A demand scenario is represented by $\boldsymbol{\xi} = (\xi_1^1, \ldots, \xi_{m_1}^1, \xi_1^2, \ldots, \xi_{m_2}^2)^\top$, where ξ_j^s denotes the demand of customer j in community s. We independently generate the customer demands according to $\xi_j^1 \sim$ Unif$[2, 2.5]$, $j = 1, \ldots, m_1$, and $\xi_j^2 \sim$ Unif$[1, 1.1]$, $j = 1, \ldots, m_2$, and consider the empirical demand data set $\Xi = \mathcal{D}_{\boldsymbol{\xi}} = \{\boldsymbol{\xi}_k\}_{k=1}^{N_{\boldsymbol{\xi}}}$ with $N_{\boldsymbol{\xi}}$ demand scenarios.

6.1.2 Algorithm Setting

We initialize Algorithm 1 with $\mathbf{w}^{(0)} = (\mathbf{x}^{\text{init}}, \mathbf{y}^{\text{init}}(\cdot))$, where $(\mathbf{x}^{\text{init}}, \mathbf{y}^{\text{init}}(\cdot))$ is the optimal solution of (7) after adding $N_{\boldsymbol{\xi}}$ additional linear constraints

$$U_1(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = U_2(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \Xi,$$

where U_s is defined in (8). These constraints require the two communities to have identical utilities in every sampled scenario, leading to zero distributional unfairness.

In line 3 of each iteration of Algorithm 1, we solve subproblem (38) using Algorithm 2. Its parameters are selected by grid search. Specifically, we choose $T_k \in \{1000, 2000, 3000\}$, $\gamma_t = t+1$, and $\theta_t = \frac{t}{t+1}$, and set $\tau_t = \frac{\tau}{t+1}$, with $\tau \in \{50, 80, 100, 150\}$ and $\eta_t = \eta(t+1)$ with $\eta \in \{5000, 6000, 8000\}$.

Table 1: Computational performance of different approaches across scenario sizes $N_{\boldsymbol{\xi}}$, with $\kappa = 1.2$

	Exact Method			Lower Bound		DCA		
$N_{\boldsymbol{\xi}}$	Runtime (s)	Objective	Optimality gap	Runtime (s)	Lower bound	Runtime (s)	Objective	Optimality gap
10	513.01	0.89	0.00%	0.96	0.89	41.81	0.90	1.47%
20	TL	NA	NA	2.08	0.91	68.71	0.92	1.46%
30	TL	NA	NA	2.94	0.91	88.20	0.92	1.44%
40	TL	NA	NA	4.22	0.89	119.06	0.91	1.54%
50	TL	NA	NA	5.56	0.89	150.93	0.91	1.52%
60	TL	NA	NA	7.65	0.90	188.42	0.91	1.50%
70	TL	NA	NA	8.68	0.90	215.59	0.91	1.53%
80	TL	NA	NA	10.69	0.90	258.29	0.92	1.53%
90	TL	NA	NA	12.83	0.90	299.73	0.92	1.52%
100	TL	NA	NA	14.65	0.90	333.17	0.92	1.50%

The value-function relaxation parameter is set to $\hat{\epsilon} = 0$. We terminate Algorithm 1 when the reduction of the objective of (9) is less than 0.5%.

Let $\mathbf{w}^{(k')} = (\mathbf{x}^{(k')}, \mathbf{y}^{(k')}(\cdot))$ denote the output of Algorithm 1. We then perform a post-processing step in which $\mathbf{x}$ is fixed at $\mathbf{x}^{(k')}$ and the recourse linear program in (7) is solved independently for each sampled scenario. This produces follower-optimal decisions $\widehat{\mathbf{y}}^{(k')}(\cdot)$. Among all parameter configurations considered in the grid search, we retain the one for which the post-processed solution $(\mathbf{x}^{(k')}, \widehat{\mathbf{y}}^{(k')}(\cdot))$ is feasible and achieves the smallest objective value for (9). This post-processed solution is used in all subsequent performance comparisons.

To obtain the exact solution of (9), we apply Gurobi to its MINLP formulation (26). For (9), functions G, Q and $\mathbf{H}$ are all linear while set $\mathcal{F}$ (21) is defined by linear and convex quadratic inequalities, so (26) is a mixed-integer convex quadratic program. We terminate Gurobi when it reaches a time limit of four hours (14,400 seconds) and report the performance of the solution at termination. The optimality gap achieved by each method is computed as $\frac{\text{Obj}-\text{LB}}{\text{Obj}} \times 100\%$, where Obj denotes the objective value of (9) at the output solution and LB denotes the lower bound of the optimal value of (9), which is set to the larger one of the two values: (1) $\text{LB}_{\text{Gurobi}}$: the lower bound produced by Gurobi at termination; (2) LB_{CP}: the lower bound obtained by the Jensen lower bound in Lemma 10 and the relaxation of the lower-level optimality constraints of (9). The details of the derivations and computations of the latter one are given in Section E.1.

6.1.3 Numerical Result

We first compare the performance of DCA and the exact method on the instances with $\kappa = 1.2$, $m_1 = m_2 = 3$, and $n = 5$. For each $N_{\boldsymbol{\xi}} \in \{10, 20, \ldots, 100\}$, we apply the exact method and DCA, and report their runtime and the achieved optimality gap in Table 1, where "TL" means Gurobi reaches the time limit. We also report the runtime for computing LB_{CP}. As shown in Table 1, the exact method fails to find a feasible solution for $N_{\boldsymbol{\xi}} \geq 30$, so its optimality gap is not available (NA) for those cases. The exact method finds a feasible solution with a large optimality gap (90.77%) when $N_{\boldsymbol{\xi}} = 20$ and it solves the problem optimally when $N_{\boldsymbol{\xi}} = 10$ but with a runtime much longer than DCA. The results show that DCA can obtain high quality solutions within a much shorter computational time. In all tested instances, the optimality gap of DCA is below 2%, indicating that DCA provides good approximate solutions for these instances.

In the second comparison, we generate instances with $N_{\boldsymbol{\xi}} = 100$, $n = 10$, and $m_1 = m_2 = 20$, for each $\kappa \in \kappa^{CP}$. We report the runtime, the final objective value, and the optimality gap of DCA in

Table 2: Computational performance for different approaches under large instances for the capacity planning problem

	Lower Bound		DCA		
κ	Runtime (s)	Lower bound	Runtime (s)	Objective	Optimality gap
1.00	2154.47	2.25	1098.45	2.25	0.00%
1.05	2411.13	1.91	1098.36	1.91	0.15%
1.10	2424.45	1.57	972.37	1.58	0.30%
1.15	2405.28	1.24	973.64	1.24	0.51%
1.20	2340.73	0.90	972.14	0.91	0.81%
1.25	2399.67	0.57	973.29	0.58	1.57%
1.30	2465.52	0.24	974.76	0.24	3.43%

Table 2. We also report the values of LB and the runtime for computing LB_{CP}. Due to the large size of these instances, the exact method fails return a feasible solution within the time limit. In contrast, DCA can obtain high quality solutions within a reasonable computational time. As κ increases, the unfairness value decreases and eventually becomes close to zero, indicating that DCA can effectively solve the proposed model under different fairness requirements. The optimality gaps of DCA are below 4% for all values of κ.

To explore the trade-off between the total cost and unfairness in (9) as κ varies, we present Figure 1, where the y-axis represents the total cost of the output solution by each method and the x-axis represents the value of the unfairness measure by the same solution. In Figure 1(a), the results are from the instance with $N_{\boldsymbol{\xi}} = 10$, $m_1 = m_2 = 3$, and $n = 5$, and in Figure 1(b), the results are from the instance with $N_{\boldsymbol{\xi}} = 100$, $m_1 = m_2 = 20$, and $n = 10$. Each marker corresponds to a value of $\kappa \in \kappa^{CP}$. To visualize the optimality gap of each method as well as the quality of LB_{CP}, we also include a set of markers whose y coordinates represent the values of LB_{CP} for each κ and x coordinates equal κc^*. By Table 1, the exact method finds the optimal solution for the instance of Figure 1(a), but it cannot find a feasible solution within the time limit for the instance of Figure 1(b) and thus is not included in the figure there. For each curve, the point with the lowest cost and the highest unfairness corresponds to the optimal solution of (7) (i.e., $\kappa = 1$), which is the same across the curves. As κ increases, the cost constraint becomes more loose, leading to solutions with lower unfairness but higher total cost. Both figures show the tightness of LB_{CP} by comparing it with the exact optimal values and the upper bounds from DCA, and both figures confirm the small optimality gaps obtained by DCA.

The trade-off in both figures has a natural interpretation in the capacity planning setting. When the decision maker focuses primarily on minimizing the total cost, capacity tends to be allocated to locations where it can be used most efficiently, for example, locations with higher demand or lower capacity and service costs. Such a cost-efficient allocation, however, may result in substantial disparities in the service received by different communities. As the fairness requirement becomes more restrictive, the decision maker must reduce these disparities by allocating capacity more evenly across communities. This may require expanding capacity in locations with relatively lower demand or higher operating costs, or reallocating resources away from the most cost-efficient locations. Consequently, improving fairness can increase the total system cost. The results in Figure 1 therefore demonstrate the trade-off between economic efficiency and equitable resource allocation.

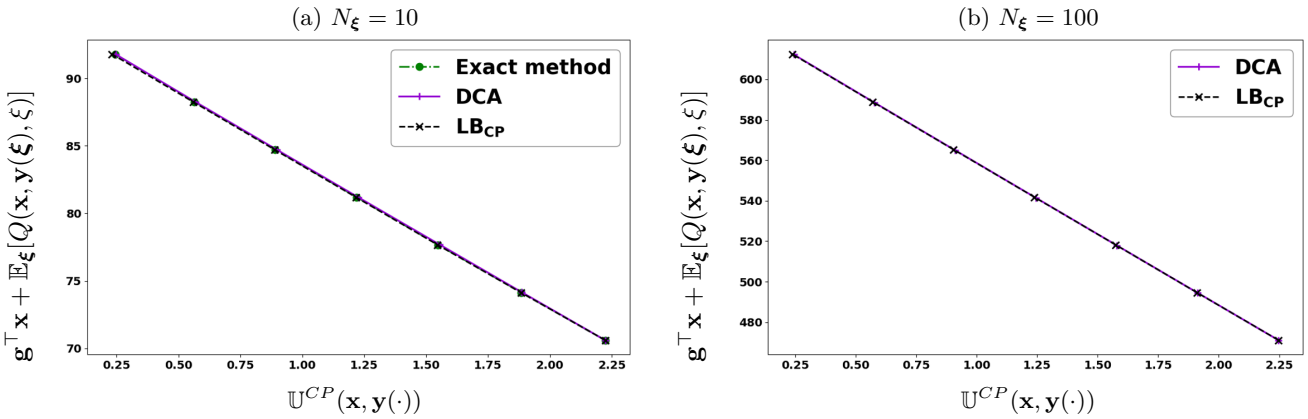


Figure 1: Cost-unfairness tradeoffs across methods as the tolerance $c = c^* \kappa$ varies, shown for two different problem sizes.

6.2 Numerical Results for Newsvendor Pricing Game

6.2.1 Parameter Settings

We consider instances of (13) with the revenue budget set to $c = \kappa c^*$, where $\kappa \in \kappa^{PG} := \{0.7, 0.8, 0.9, 0.95, 0.96, 0.98\}$ and c^* denotes the optimal objective value of the cost-only pricing game (10), which is a mixed integer quadratic program and solved by Gurobi. In all, instances, $c^* < 0$ and thus κ controls the additional reward the wholesaler gives up in pursuit of distributional fairness. We consider $S = n = 7$ communities, set $c_i = \frac{2i}{n}$ for $r_i = h_i = b_i = 1$, $i = 1, \ldots, n$, and $N_{\boldsymbol{\zeta}} = 100$. A demand scenario is represented by $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)^\top$, where ξ_j^s denotes the demand of customer j in community s. We independently generate each ζ_i from a truncated Gaussian distribution over $[20, 60]$ with mean equal to 40 and standard deviation equal to 10. Finally, the empirical demand data set $\mathrm{Z} = \mathcal{D}_{\boldsymbol{\zeta}} = \{\boldsymbol{\zeta}_k\}_{k=1}^{N_{\boldsymbol{\zeta}}}$ consists of $N_{\boldsymbol{\zeta}}$ scenarios.

6.2.2 Algorithm Setting

We initialize Algorithm 1 with $\mathbf{w}^{(0)} = (\mathbf{x}^{\text{init}}, \mathbf{y}^{\text{init}})$, where $(\mathbf{x}^{\text{init}}, \mathbf{y}^{\text{init}})$ is the optimal solution of (10) solved by Gurobi. The time for computing $\mathbf{w}^{(0)}$ will be included in the total runtime of the DCA method for a fair comparison.

In line 3 of each iteration of Algorithm 1, we solve subproblem (38) using Algorithm 2. Its parameters are selected by grid search. Specifically, we choose $T_k \in \{1000, 2000\}$, $\gamma_t = t + 1$, and $\theta_t = \frac{t}{t+1}$, and set $\tau_t = \frac{\tau}{t+1}$, with $\tau \in \{500, 1000\}$ and $\eta_t = \eta(t+1)$ with $\eta \in \{10, 300, 500, 1000\}$. The value-function relaxation parameter is set to $\hat{\epsilon} = 0.01$. The proximal parameter ρ is selected from $\{0, 400\}$. We terminate Algorithm 1 when the reduction of the objective of (13) is less than 0.5%.

After obtaining the solution $\mathbf{w}^{(k')} = (\mathbf{x}^{(k')}, \mathbf{y}^{(k')}(\cdot))$ from Algorithm 1, we apply an alternating projection procedure to recover a feasible solution for the original nonconvex problem. Specifically, we alternately project the current solution onto each of the two constraints in (13), using the output of each projection as the input to the next. Although the projection to their intersection is non-trivial, the projection to each set can be solved in polynomial time. The details of how the projections are computed are given in the Appendix G. This process is repeated until the violation of revenue constraint and bilevel constraint are smaller than 0.0001. We then denote the solution after this post-processing step by $(\hat{\mathbf{x}}^{(k')}, \hat{\mathbf{y}}^{(k')})$. Among all parameter configurations considered in the grid search above, we retain the one for which $(\hat{\mathbf{x}}^{(k')}, \hat{\mathbf{y}}^{(k')})$ achieves the smallest objective value for (13). This post-processed solution is used in all subsequent performance comparisons.

Table 3: Computational performance for different approaches under large instances for newsvendor pricing game

	Exact Method				DCA		
κ	Runtime (s)	Objective	Objective at DCA time	Optimality gap at DCA time	Runtime (s)	Objective	Optimality gap
0.70	2598.19	1.50	7.64	80.36%	72.29	1.61	6.60%
0.80	817.37	1.58	4.82	67.31%	144.77	1.67	5.64%
0.90	761.56	1.59	2.46	35.40%	227.00	1.70	6.53%
0.95	TL	3.32	NA	NA	414.29	3.68	9.70%
0.96	3626.07	3.86	NA	NA	596.85	4.24	8.98%
0.98	TL	5.53	NA	NA	309.14	6.09	10.55%

To obtain the exact solution of (13), we first reformulate the lower level problem of (13) by introducing auxiliary binary variables and linear constraints to remove the $(\cdot)_+$ functions, which Gurobi cannot directly process. Then the lower level decision variable $\mathbf{y}$ in (13) will be augmented with those auxiliary binary variables. Then we apply Gurobi to its MINLP formulation (26) with the augmented $\mathbf{y}$ and the additional constraints. We present the details of this reformulation in the Appendix F. For (13), function G is quadratic and Q and $\mathbf{H}$ are all linear after introducing the auxiliary variables. Additionally, set $\mathcal{F}'$ (21) is defined by linear and convex quadratic inequalities, so (26) is a mixed-integer convex quadratic program.

However, as shown later, Gurobi fails to produce even a feasible solution (13) to when applied to (26) as described above because of the large number of integer variables involved in $\mathcal{F}'$. To improve the efficiency of Gurobi, we propose an alternative exact method based on a reformulation of the objective function of (13) using the monotonicity of the utility function in (12) in the demand ζ_s. We derive a new epigraph of the Wasserstein unfairness measure for this stronger benchmark, called $\mathcal{F}'^{PG}$. We provides the details of this reformulation in Appendix F.1. As this problem-specific single-level reformulation computationally outperforms the reformulation based on (26) (see results in Table 4), we utilize this stronger benchmark in reporting the performance of the exact method in our computational study.

We tried to compute the lower bound LB_{PG} in a similar way to the capacity planning problem. However, LB_{PG} obtained for each instance is very loose compared with lower bounds obtained by Gurobi. Therefore, it is not used to construct LB.

6.2.3 Numerical Results

We first compare the performances of the exact formulation solved by Gurobi with that of DCA in Table 3. Given the results in Table 4, we only present the results of the exact method based on (96). For each method, we report the objective value (UB) at the output solution, the optimality gap, and the runtime. To demonstrate the merit of DCA when Gurobi can find the optimal solutions, we also terminate Gurobi at the same time when DCA is terminated and report the objective value and the optimality gap of the output solution of Gurobi.

The results show that Gurobi finds the optimal solution but needs longer runtime, while DCA finds a good solution is significantly less runtime with most of the optimality gap no more than 10%. The table also shows that, when terminated at the same time, DCA achieves much better solutions than Gurobi. For example, when $\kappa = 0.96$, Gurobi requires 3626.07 seconds, whereas DCA takes only 596.85 seconds. For several instances, Gurobi reaches the prescribed time limit, while DCA still obtains a solution within a few minutes. These results demonstrate that DCA provides a

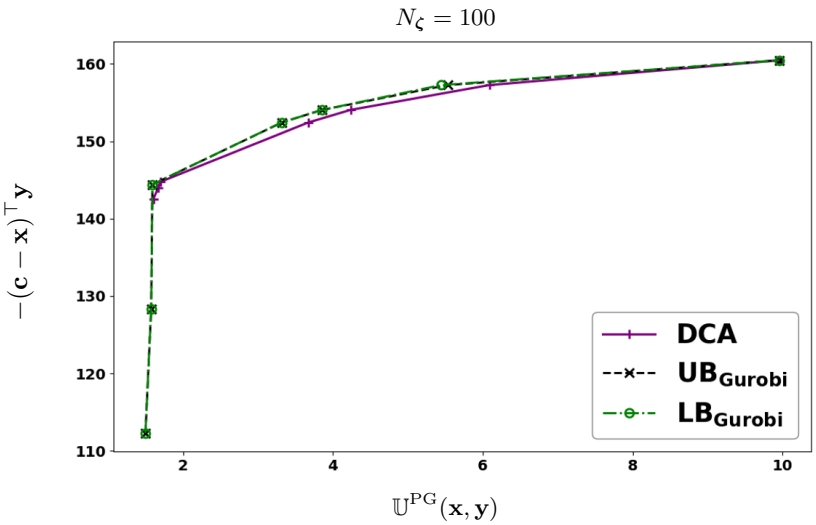


Figure 2: Negative cost-unfairness tradeoffs across methods as the tolerance $c = c^* \kappa$ varies, shown for one problem sizes.

favorable trade-off between solution quality and computational efficiency for large-scale instances of the newsvendor pricing game.

We also examine how the revenue-unfairness trade-off changes as κ decreases for the different solution approaches. The results are presented in Figure 2, where the y-axis represents the total expected revenue and the x-axis represents the maximum unfairness across retailers. For each approach, the curve starts from the least fairness-constrained solution, located in the upper-right region of the figure, which achieves both high revenue and high unfairness. As κ decreases, the fairness requirement becomes more restrictive, leading to solutions with lower unfairness but also lower revenue. This trade-off has a natural interpretation in the wholesaler-retailer setting. When the wholesaler focuses primarily on maximizing total revenue, it may allocate products or set decisions in a way that favors retailers with higher demand, higher profitability, or more advantageous market conditions. As a result, some retailers may receive substantially more favorable outcomes than others, which increases the disparity across retailers. In contrast, imposing a stronger fairness requirement forces the wholesaler to reduce these disparities and provide more balanced outcomes across retailers. Doing so may require shifting resources or decisions away from the most profitable retailers toward less profitable ones, thereby sacrificing part of the total revenue. Therefore, as κ decreases, the solutions move toward lower unfairness at the expense of revenue, illustrating the fundamental trade-off between economic efficiency and fairness.

7 Conclusion

This paper develops a distributionally fair bilevel optimization framework that reduces disparities across community utility distributions while maintaining the leader's expected cost within a prescribed threshold and accounting for the follower's optimal responses. Under empirical distributions, we derive an exact MINLP reformulation and lower-bounding schemes to assess solution quality. For larger instances, we develop a DC reformulation and an inexact DCA that accommodates nonsmooth lower-level problems with potentially nonunique optimal responses. Under the stated assumptions, we establish guarantees for obtaining nearly ϵ-KKT points and characterize the overall iteration complexity. Computational experiments on capacity planning and newsvendor pricing games demonstrate that the DC algorithm obtains high-quality solutions in substantially less time

than the exact formulations solved by a general-purpose solver. The results also illustrate how leader decisions can reduce distributional disparities through follower responses, revealing trade-offs between economic performance and fairness.

A Proofs of Technical Results

In this section, we present the proofs of the technical results in the main paper.

A.1 Proof of Lemma 2

Proof of Lemma 2. By the definition of the graph $\Omega_s(k)$, we have

$$\Omega_s(k) = \left\{ (\mathbf{x}, \mathbf{y}(\cdot), t_{ks}) \in \mathcal{X} \times \mathcal{Y}^{N_\xi} \times \mathbb{R} : \begin{array}{l} \upsilon_{\ell ks} \in \{0,1\}, z_{\ell ks} \in \{0,1\}, \hat{t}_{\ell ks} \in \mathbb{R}, \text{ for } \ell \in \mathcal{L}, \\ \upsilon_{\ell ks} \le z_{\ell ks}, \text{ for } \ell \in \mathcal{L}, \sum_{\ell \in \mathcal{L}} z_{\ell ks} = k, \sum_{\ell \in \mathcal{L}} \upsilon_{\ell ks} = 1, \\ t_{ks} = \sum_{\ell \in \mathcal{L}} \upsilon_{\ell ks} u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot))), \\ t_{ks} \ge u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) - 2M_k^s(1 - z_{\ell ks}), \\ t_{ks} \le u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)) + 2M_k^s z_{\ell ks}, \end{array} \right\}. \quad (54)$$

The conclusion is then obtained by setting $\hat{t}_{\ell ks} = \upsilon_{\ell ks} u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)))$ and linearizing the bilinear term of $\upsilon_{\ell ks}$ and $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot)))$ using the equivalent McCormick linear representation. □

A.2 Proof of Theorem 7

Proof of Theorem 7. Under Assumption 4, the follower problem is convex and satisfies Slater's condition, and its optimal value is attained. Therefore, the KKT conditions (22)–(24) are necessary and sufficient for the optimality of $\mathbf{y}(\boldsymbol{\xi})$ for every $\boldsymbol{\xi} \in \Xi$.

We next show that the Big-M constraints (25), together with primal and dual feasibility, provide an exact representation of complementary slackness.

First, consider any follower-optimal solution $\mathbf{y}(\boldsymbol{\xi})$. By Assumption 6, there exists an associated KKT multiplier $\boldsymbol{\lambda}(\boldsymbol{\xi}) = (\lambda_1(\boldsymbol{\xi}), \ldots, \lambda_r(\boldsymbol{\xi}))^\top$ satisfying the specified bounds. For $j = 1, \ldots, r$, define

$$\mathbf{w}(\boldsymbol{\xi}) = (w_1(\boldsymbol{\xi}), \ldots, w_r(\boldsymbol{\xi}))^\top \text{ and } w_j(\boldsymbol{\xi}) = \begin{cases} 1, & \lambda_j(\boldsymbol{\xi}) > 0, \\ 0, & \lambda_j(\boldsymbol{\xi}) = 0. \end{cases}$$

If $w_j(\boldsymbol{\xi}) = 1$, complementary slackness implies $H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0$. If $w_j(\boldsymbol{\xi}) = 0$, then $\lambda_j(\boldsymbol{\xi}) = 0$. Together with the bounds in Assumption 6, these observations show that (25) holds.

Conversely, suppose that (25) and (24) hold. For each $j = 1, \ldots, r$, if $w_j(\boldsymbol{\xi}) = 0$, then $\lambda_j(\boldsymbol{\xi}) = 0$. If $w_j(\boldsymbol{\xi}) = 1$, primal feasibility and the second inequality in (25) imply $H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0$. Therefore, $\lambda_j(\boldsymbol{\xi}) H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0$ for $j = 1, \ldots, r$. Thus, the complementary slackness condition (22) holds.

Consequently, under Assumptions 4 and 6, the constraints (23), (24), and (25) provide an exact representation of the follower's optimality conditions.

Finally, Lemma 3 establishes that $\mathcal{F}'$ is an equivalent representation of $\mathcal{F}$. Since the upper-level objective and expected-cost constraint remain unchanged, problems (16) and (26) have identical feasible sets after projecting out the auxiliary variables. Hence, they have the same optimal objective value. □

A.3 Proof of Proposition 9

Proof of Proposition 9. We first show that the lower-level optimality conditions admit a mixed-integer linear representation. Define

$$q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) := \mathbb{E}_{\boldsymbol{\zeta}} \left[Q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) \right].$$

Because the expectation is a finite weighted sum, q is convex piecewise affine. We may therefore write

$$q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = \max_{i \in \mathcal{I}_0} \left\{ \mathbf{a}_{0i}^\top \mathbf{y} + \mathbf{b}_{0i}^\top \mathbf{x} + d_{0i} \right\}, \quad H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) = \max_{i \in \mathcal{I}_j} \left\{ \mathbf{a}_{ji}^\top \mathbf{y} + \mathbf{b}_{ji}^\top \mathbf{x} + d_{ji} \right\}, \quad j = 1, \ldots, r,$$

where every index set $\mathcal{I}_j$ is finite.

For $j = 0, \ldots, r$, we introduce a continuous variable t_j representing the corresponding maximum and binary variables z_{ji} identifying selected active affine pieces. Using valid constants M_{ji}, impose

$$\mathbf{a}_{ji}^\top \mathbf{y}(\boldsymbol{\xi}) + \mathbf{b}_{ji}^\top \mathbf{x} + d_{ji} \le t_j, \tag{55}$$

$$t_j \le \mathbf{a}_{ji}^\top \mathbf{y}(\boldsymbol{\xi}) + \mathbf{b}_{ji}^\top \mathbf{x} + d_{ji} + M_{ji}(1 - z_{ji}), \tag{56}$$

$$\sum_{i \in \mathcal{L}_j} z_{ji} \ge 1, \qquad z_{ji} \in \{0, 1\}. \tag{57}$$

These constraints ensure that $t_0 = q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ and $t_j = H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ for $j \ge 1$, and that any piece with $z_{ji} = 1$ is active. Multiple active pieces may be selected, which is necessary to represent subgradients at nondifferentiable points.

The subdifferential of a convex piecewise affine function is the convex hull of the gradients of its active pieces. Consequently, any $\mathbf{g}_0 \in \partial_{\mathbf{y}} q(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ and any $\mathbf{g}_j \in \partial_{\mathbf{y}} H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ for $j = 1, \ldots, r$ can be represented as

$$\mathbf{g}_0 = \sum_{i \in \mathcal{I}_0} \theta_{0i} \mathbf{a}_{0i}, \qquad \sum_{i \in \mathcal{I}_0} \theta_{0i} = 1, \quad 0 \le \theta_{0i} \le z_{0i},$$

$$\mathbf{g}_j = \sum_{i \in \mathcal{I}_j} \theta_{ji} \mathbf{a}_{ji}, \qquad \sum_{i \in \mathcal{I}_j} \theta_{ji} = 1, \quad 0 \le \theta_{ji} \le z_{ji},$$

For each constraint j, the stationarity condition (23) becomes (27), which involves a vector in $\lambda_j(\boldsymbol{\xi}) \partial_{\mathbf{y}} H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$. To avoid multiplying $\lambda_j(\boldsymbol{\xi})$ by θ_{ji}, we introduce nonnegative variables ρ_{ji}'s. Then any vector in $\lambda_j(\boldsymbol{\xi}) \partial_{\mathbf{y}} H_j(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi})$ can be written as $\sum_{i \in \mathcal{I}_j} \rho_{ji} \mathbf{a}_{ji}$ with ρ_{ji}'s satisfying

$$\sum_{i \in \mathcal{I}_j} \rho_{ji} = \lambda_j(\boldsymbol{\xi}), \qquad 0 \le \rho_{ji} \le M z_{ji},$$

where M is from Assumption 6. Thus, (27) is equivalent to the linear equality

$$\sum_{i \in \mathcal{I}_0} \theta_{0i} \mathbf{a}_{0i} + \sum_{j=1}^{r} \sum_{i \in \mathcal{I}_j} \rho_{ji} \mathbf{a}_{ji} = 0.$$

Primal feasibility is represented by $t_j \le 0$, $j = 1, \ldots, r$. Dual feasibility and complementary slackness can be represented by

$$0 \le \lambda_j(\boldsymbol{\xi}) \le M w_j(\boldsymbol{\xi}), \qquad -M(1 - w_j(\boldsymbol{\xi})) \le t_j \le 0, \qquad w_j(\boldsymbol{\xi}) \in \{0, 1\},$$

where M is from Assumption 6. Hence, all lower-level optimality conditions admit a mixed-integer linear representation. Repeating this construction for every scenario $\boldsymbol{\xi}$ yields a finite system.

Next, the graph of a continuous piecewise affine function with finitely many pieces is a finite union of polyhedra. Binary variables can therefore select an affine region and give an exact mixed-integer linear representation of the function's value over the bounded decision domain. Applying this construction to $G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})$, and expressing its expectation as a finite weighted sum, yields a mixed-integer linear representation of the upper-level cost constraint. The same argument applies to each sampled utility $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot))$.

Finally, according to the representation of the graph $\Omega_s(k)$ in (54), the constraints defining $\Omega_s(k)$ in (54) become mixed-integer linear constraints after replacing the sampled utilities $u_{s,\ell}(\mathbf{x}, \mathbf{y}(\cdot))$ by their mixed-integer linear representations mentioned above. The remaining Wasserstein epigraph constraints are

$$\frac{1}{N}\sum_{k=1}^{N}(t_{ks} - t_{ks'})^2 \le \nu, \qquad s, s' \in \mathcal{S}, \quad s \neq s',$$

which are convex quadratic inequalities. Therefore, the resulting formulation has a linear objective, linear and convex quadratic constraints, and continuous and binary variables. This establishes the claimed mixed-integer convex quadratically constrained representation.

□

A.4 Proof of Proposition 14

Proof of Proposition 14. For simplicity of notation, we write $\mathbf{y}(\cdot)$ just as $\mathbf{y}$ in this proof. For each $s, s' \in \mathcal{S}$ with $s < s'$ and $\ell \in \mathcal{L}$, define

$$D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) := \sum_{\ell' \in \mathcal{L}} d^{+}_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}).$$

Then, for every ℓ, ℓ', we can rewrite

$$\left[u_{s,\ell}(\mathbf{x}, \mathbf{y}) - u_{s',\ell'}(\mathbf{x}, \mathbf{y})\right]^2 = D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) - R_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}), \tag{58}$$

where

$$R_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}) := \left[D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) - d^{+}_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y})\right] + d^{-}_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}).$$

Since

$$D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) - d^{+}_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}) = \sum_{\bar{\ell} \neq \ell'} d^{+}_{ss',\ell\bar{\ell}}(\mathbf{x}, \mathbf{y}),$$

the function $R_{ss',\ell\ell'}$ is convex as a sum of convex functions.

Substituting (58) into (15), we obtain

$$\begin{aligned}
\mathbb{W}_2^2(\mathbb{P}_{U_s}, \mathbb{P}_{U_{s'}}) &= \min_{\boldsymbol{\pi} \in \Pi} \sum_{\ell \in \mathcal{L}} \sum_{\ell' \in \mathcal{L}} \pi_{\ell\ell'} \left[D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) - R_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y})\right] \\
&= \frac{1}{N}\sum_{\ell \in \mathcal{L}} D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y}) - \max_{\boldsymbol{\pi} \in \Pi} \sum_{\ell \in \mathcal{L}} \sum_{\ell' \in \mathcal{L}} \pi_{\ell\ell'} R_{ss',\ell\ell'}(\mathbf{x}, \mathbf{y}),
\end{aligned} \tag{59}$$

where the second equality follows from the marginal constraint on $\boldsymbol{\pi} \in \Pi$. Therefore, defining

$$A_{ss'}(\mathbf{x}, \mathbf{y}) := \frac{1}{N}\sum_{\ell \in \mathcal{L}} D^{+}_{ss',\ell}(\mathbf{x}, \mathbf{y})$$

and

$$B_{ss'}(\mathbf{x},\mathbf{y}) := \max_{\boldsymbol{\pi}\in\Pi} \sum_{\ell\in\mathcal{L}} \sum_{\ell'\in\mathcal{L}} \pi_{\ell\ell'} R_{ss',\ell\ell'}(\mathbf{x},\mathbf{y}),$$

we have the DC representation

$$\mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}) = A_{ss'}(\mathbf{x},\mathbf{y}) - B_{ss'}(\mathbf{x},\mathbf{y}).$$

Indeed, $A_{ss'}$ is convex by construction. Moreover, for each fixed $\boldsymbol{\pi}\in\Pi$, the function

$$\sum_{\ell\in\mathcal{L}} \sum_{\ell'\in\mathcal{L}} \pi_{\ell\ell'} R_{ss',\ell\ell'}(\mathbf{x},\mathbf{y})$$

is convex because $\boldsymbol{\pi}\geq 0$ and each $R_{ss',\ell\ell'}$ is convex. Hence $B_{ss'}$ is convex as the pointwise maximum of convex functions.

Consequently, each empirical squared Wasserstein distance $\mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}})$ is a DC function of $(\mathbf{x},\mathbf{y})$. The squared distributional unfairness measure can then be written as

$$\mathbb{U}(\mathbf{x},\mathbf{y}) := \max_{s,s'\in\mathcal{S},s<s'} \mathbb{W}_2^2(\mathbb{P}_{U_s},\mathbb{P}_{U_{s'}}) = \max_{s,s'\in\mathcal{S},s<s'} \left\{A_{ss'}(\mathbf{x},\mathbf{y}) - B_{ss'}(\mathbf{x},\mathbf{y})\right\}.$$

Because the maximum is taken over finitely many community pairs, $\mathbb{U}$ is also DC. In fact,

$$\mathbb{U}(\mathbf{x},\mathbf{y}) = \max_{s,s'\in\mathcal{S},s<s'} \left\{ A_{ss'}(\mathbf{x},\mathbf{y}) + \sum_{r,r'\in\mathcal{S},r<r',\,(r,r')\neq(s,s')} B_{rr'}(\mathbf{x},\mathbf{y}) \right\} - \sum_{r,r'\in\mathcal{S},r<r'} B_{rr'}(\mathbf{x},\mathbf{y}). \tag{60}$$

The first term in (60) is $\mathbb{U}^+$, which is convex as the pointwise maximum of finitely many convex functions, and the second term is $\mathbb{U}^-$, which is also convex as a finite sum of convex functions. Therefore, $\mathbb{U}$ is DC. □

B Computing Value Functions and Their Subgradients

To solve DC reformulation (32) by a first-order method, one needs to compute the value functions $\phi^+(\mathbf{x},\boldsymbol{\xi})$ and $\phi^-(\mathbf{x},\boldsymbol{\xi})$ and their subgradients with respect to $\mathbf{x}$. In this section, we will illustrate how they can be computed easily for the two applications from Section 2.

Capacity planning problem. Consider the distributionally fair capacity planning problem in (9). The lower-level problem is a linear program for each $\boldsymbol{\xi}\in\Xi$, where the leader's decision $\mathbf{x}$ only appears on the right-hand side of the constraints. Therefore, the lower-level value function $\phi(\mathbf{x},\boldsymbol{\xi})$ is a convex function of $\mathbf{x}$, which satisfies Assumption 18 with $\phi^+(\mathbf{x},\boldsymbol{\xi}) = \phi(\mathbf{x},\boldsymbol{\xi})$ and $\phi^-(\mathbf{x},\boldsymbol{\xi}) = 0$.

For any $\mathbf{x}$ and $\boldsymbol{\xi}$, the lower-level problem in (9) is a transportation-type linear program. By introducing unmet-demand variables

$$u_j^s = \xi_j^s - \sum_{i=1}^n y_{ij}^s, \qquad s\in\mathcal{S}, \quad j=1,\ldots,m_s,$$

the lower-level problem can be equivalently written as

$$\begin{aligned} \phi(\mathbf{x},\boldsymbol{\xi}) = \phi^+(\mathbf{x},\boldsymbol{\xi}) = \min_{\mathbf{y}\geq 0,\mathbf{u}\geq 0} \quad & \sum_{i=1}^n \sum_{s\in\mathcal{S}} \sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s + \sum_{s\in\mathcal{S}} \sum_{j=1}^{m_s} p_s u_j^s \\ \text{s.t.} \quad & \sum_{s\in\mathcal{S}} \sum_{j=1}^{m_s} y_{ij}^s \leq x_i o_i, \quad i=1,\ldots,n, \\ & \sum_{i=1}^n y_{ij}^s + u_j^s = \xi_j^s, \quad s\in\mathcal{S}, \quad j=1,\ldots,m_s. \end{aligned} \tag{61}$$

This formulation is a minimum-cost flow problem on a bipartite network with production sites on one side and customers on the other, together with dummy arcs that represent unmet demand. Therefore, for each fixed $(\mathbf{x}, \boldsymbol{\xi})$, an optimal follower decision $\mathbf{y}(\boldsymbol{\xi})$ can be computed in polynomial time using standard minimum-cost flow algorithms (Ahuja et al. 1993).

Moreover, the dual variables associated with the production-capacity constraints can be obtained together with an optimal primal solution. Specifically, let $\lambda_i \geq 0$ denote the Lagrange multiplier associated with the capacity constraint at production site i, and let μ_j^s denote the free dual variable associated with the demand-balance equality for customer j in community s. The dual of (61) is

$$\begin{aligned} \phi^+(\mathbf{x}, \boldsymbol{\xi}) = \max_{\boldsymbol{\lambda} \geq 0, \boldsymbol{\mu}} \quad & \sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} \mu_j^s \xi_j^s - \sum_{i=1}^{n} \lambda_i x_i o_i \\ \text{s.t.} \quad & \mu_j^s \leq c_{ij}^s + \lambda_i, \quad i = 1, \ldots, n, \quad s \in \mathcal{S}, \quad j = 1, \ldots, m_s, \\ & \mu_j^s \leq p_s, \quad s \in \mathcal{S}, \quad j = 1, \ldots, m_s. \end{aligned} \tag{62}$$

Let $(\boldsymbol{\lambda}(\boldsymbol{\xi}), \boldsymbol{\mu}(\boldsymbol{\xi}))$ be any optimal solution of the dual problem (62). Then, by Danskin's theorem, we have

$$-\mathbf{o} \odot \boldsymbol{\lambda}(\boldsymbol{\xi}) \in \partial_{\mathbf{x}} \phi^+(\mathbf{x}, \boldsymbol{\xi}),$$

where $\odot$ means componentwise (Hadamard) product. Thus, the subgradient of $\phi^+(\mathbf{x}, \boldsymbol{\xi})$ required by a first-order method can be obtained from the dual solution of the same minimum-cost flow problem. In practice, such dual variables are available from standard LP solvers and from polynomial-time primal-dual implementations of minimum-cost flow algorithms, including network simplex methods; see, for example, Ahuja et al. (1993).

Newsvendor pricing game. Consider the distributionally fair newsvendor pricing game (13). Since there is no $\boldsymbol{\xi}$, we just write $\phi(\mathbf{x}, \boldsymbol{\xi})$ as $\phi(\mathbf{x})$ and the set of scenarios $\mathcal{L}$ is essentially reduced to $\{1, \ldots, N_{\boldsymbol{\zeta}}\}$. For $s \in \mathcal{S}$ and $\ell \in \mathcal{L}$, let $\zeta_{\ell,s}$ be the demand of community s under scenario $\boldsymbol{\zeta}_\ell$ and $\zeta_{(\ell),s}$ be ℓth smallest demand among the $N_{\boldsymbol{\zeta}}$ scenarios, which means $\zeta_{(1),s} \leq \zeta_{(2),s} \leq \cdots \leq \zeta_{(N_{\boldsymbol{\zeta}}),s}$.

According to the lower-level problem in (13), we have For a fixed wholesale price vector $\mathbf{x}$, the lower-level problem in the newsvendor pricing game is separable across retailers:

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(x_i), \tag{63}$$

where

$$\phi_i(x_i) := \min_{\underline{y}_i \leq y_i \leq \overline{y}_i} \frac{1}{N_{\boldsymbol{\zeta}}} \sum_{\ell=1}^{N_{\boldsymbol{\zeta}}} \left[b_i (\zeta_{\ell,i} - y_i)_+ + (h_i + r_i)(y_i - \zeta_{\ell,i})_+ + (x_i - r_i) y_i \right]. \tag{64}$$

For each fixed y_i, the objective in (64) is affine in x_i, so $\phi_i(x_i)$ is concave in x_i. In particular, $\phi(\mathbf{x})$ satisfies Assumption 18 with $\phi^+(\mathbf{x}) = 0$ and $\phi^-(\mathbf{x}) = -\phi(\mathbf{x})$.

The objective function in (64) is a standard cost function in a newsvendor problem. It is then well known that an optimal solution to (64) is

$$y_i^*(x_i) = \zeta_{\ell_i(x_i),i}, \quad \ell_i(x_i) := \max \left\{ 1, \left\lceil N_{\boldsymbol{\zeta}} \frac{b_i + r_i - x_i}{b_i + h_i + r_i} \right\rceil \right\}$$

for $i = 1, \ldots, n$. Here, $(b_i + r_i - x_i)/(b_i + h_i + r_i)$ is the critical ratio, and it always lies in $[0, 1]$ because of the bounds of x_i defined in (11). Then

$$\mathbf{y}^*(\mathbf{x}) = (y_1^*(x_1), \ldots, y_n^*(x_n))^\top$$

is an optimal solution of the lower-level problem (63), which can be easily computed by sorting the empirical demand samples for each retailer and identifying the $\ell_i(x_i)$th smallest sample.

By Danskin's theorem, $-y_i^*(x_i) \in \partial[-\phi_i(x_i)]$. and thus

$$-\mathbf{y}^*(\mathbf{x}) \in \partial[-\phi(\mathbf{x})] = \partial\phi^-(\mathbf{x}).$$

C Verification of Assumption 23 for Applications (9) and (13)

We verify Assumption 23 for the empirical versions of both applications. In particular, we establish its parameter-dependent Slater condition B. For all sufficiently small $\delta > 0$, we choose

$$\epsilon_\delta = 2\delta, \qquad \nu_\delta = \kappa\delta, \tag{65}$$

with a constant $\kappa > 0$ independent of δ and of the base point $\mathbf{w}$. We show that, whenever

$$f_i(\mathbf{w}) - \delta \le \epsilon_\delta^2 = 4\delta^2, \qquad i \in [m], \tag{66}$$

there exists $\mathbf{v} \in \mathrm{int}(\mathcal{W})$ such that

$$g_i(\mathbf{v}; \mathbf{w}) - \delta \le -\nu_\delta, \qquad i \in [m]. \tag{67}$$

Thus $\nu_\delta = \Theta(\delta)$ for both applications, and no separate relaxation of the original DC constraints is needed for this verification.

Assumption 23A can be enforced by adding $\frac{\mu}{2}\|\mathbf{w}\|^2$ to both convex components of $f_0 = f_0^+ - f_0^-$ for any $\mu > 0$. The objective remains unchanged, and the new components are μ-strongly convex. We verify the remaining conditions below.

C.1 Capacity planning problem

Consider (9). Let $\boldsymbol{\xi}_k = (\xi^s_{k,j})_{s,j}$, $k = 1, \dots, N_{\boldsymbol{\xi}}$, be the empirical demand scenarios, where $\xi^s_{k,j}$ denotes the demand of customer j in community s. Write $\mathbf{y}(\boldsymbol{\xi}_k) = (y^s_{ij}(\boldsymbol{\xi}_k))_{i,j,s}$ for the corresponding shipment decisions. For $\boldsymbol{\xi} \in \Xi$, define

$$q^{CP}(\mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) := \sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} c^s_{ij} y^s_{ij}(\boldsymbol{\xi}) + \sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} p_s \left(\xi^s_j - \sum_{i=1}^{n} y^s_{ij}(\boldsymbol{\xi})\right), \tag{68}$$

$$\phi^{CP}(\mathbf{x}, \boldsymbol{\xi}) := \min_{\mathbf{y}(\boldsymbol{\xi})\ge 0} \left\{ q^{CP}(\mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}) : \begin{array}{ll} \displaystyle\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} y^s_{ij}(\boldsymbol{\xi}) \le x_i o_i, & \forall i, \\ \displaystyle\sum_{i=1}^{n} y^s_{ij}(\boldsymbol{\xi}) \le \xi^s_j, & \forall s, j \end{array} \right\}. \tag{69}$$

The empirical upper-level cost is

$$C^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) := \mathbf{g}^\top \mathbf{x} + \frac{1}{N_{\boldsymbol{\xi}}} \sum_{k=1}^{N_{\boldsymbol{\xi}}} q^{CP}(\mathbf{y}(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k).$$

The capacity constraints retained explicitly in the DC formulation are

$$H^{CP}_{i,k}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}_k)) := \sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} y^s_{ij}(\boldsymbol{\xi}_k) - x_i o_i, \quad i = 1, \dots, n, \quad k = 1, \dots, N_{\boldsymbol{\xi}}.$$

Following the main text, incorporate the other lower-level feasibility constraints into

$$\mathcal{W}^{CP} := \Big\{(\mathbf{x}, \mathbf{y}(\cdot)) : \mathbf{x} \in [0,1]^n, \quad \mathbf{y}(\boldsymbol{\xi}_k) \geq 0, \\ \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi}_k) \leq \xi_{k,j}^s, \quad \forall s, j, k\Big\}. \tag{70}$$

Thus the explicit DC constraints are

$$\begin{aligned} &C^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) - c \leq 0, \\ &q^{CP}(\mathbf{y}(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k) - \phi^{CP}(\mathbf{x}, \boldsymbol{\xi}_k) \leq 0, \quad k = 1, \ldots, N_{\boldsymbol{\xi}}, \\ &H_{i,k}^{CP}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}_k)) \leq 0, \quad i = 1, \ldots, n, \quad k = 1, \ldots, N_{\boldsymbol{\xi}}. \end{aligned}$$

Proposition 29 (Verification for capacity planning)**.** *Suppose that the empirical demands and cost parameters are finite, $\mathcal{W}^{CP}$ has nonempty interior, and $c > c_{CP}^*$, where c_{CP}^* is the optimal value of the empirical fairness-unaware capacity-planning problem (7). Then Assumption 23A–E holds, with $\epsilon_\delta = 2\delta$ and $\nu_\delta = \delta/2$ in condition B for all sufficiently small $\delta > 0$. The result holds for any subgradients selected by the DCA from the finite-valued polyhedral extensions specified below.*

Proof. Assumption 23A follows from the common quadratic shift above. The polytope $\mathcal{W}^{CP}$ is convex and compact, since nonnegativity and customer demand constraints bound all shipment variables. The convex components of the empirical fairness objective are Lipschitz continuous on this set by Remark 15. The upper-level cost, capacity constraints, and q^{CP} are affine.

For each scenario $\boldsymbol{\xi}_k$, the value function $\phi^{CP}(\cdot, \boldsymbol{\xi}_k)$ is finite and convex polyhedral on $[0,1]^n$. Hence, it agrees on this box with the pointwise maximum of finitely many affine functions and admits a finite-valued, globally Lipschitz continuous convex polyhedral extension. We use this extension in the DC decomposition; it does not change the model on $\mathcal{W}^{CP}$ and ensures bounded subgradients even when some $x_i = 0$. Thus Assumptions 23C–D hold.

For Assumption 23E, choose an optimal fairness-unaware solution $\mathbf{w}^* = (\mathbf{x}^*, \mathbf{y}^*(\cdot))$, with each $\mathbf{y}^*(\boldsymbol{\xi}_k)$ follower-optimal at $\mathbf{x}^*$. Then all capacity and value-function constraints hold and $C^{CP}(\mathbf{w}^*) = c_{CP}^* < c$. Hence $\mathbf{w}^*$ is a feasible initial point.

We now verify Assumption 23B. For each k, fix a finite-valued polyhedral extension of the value function to $\mathbb{R}^n$ and a finite affine representation

$$\phi^{CP}(\mathbf{x}, \boldsymbol{\xi}_k) = \max_{\ell \in J_k} \left\{\mathbf{a}_{k\ell}^\top \mathbf{x} + b_{k\ell}\right\}, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{71}$$

where the left-hand side denotes this extension outside $[0,1]^n$. For each $\mathbf{x} \in [0,1]^n$, define its active index set

$$A_k(\mathbf{x}) := \{\ell \in J_k : \mathbf{a}_{k\ell}^\top \mathbf{x} + b_{k\ell} = \phi^{CP}(\mathbf{x}, \boldsymbol{\xi}_k)\}.$$

At an arbitrary base point $\mathbf{w} = (\mathbf{x}, \mathbf{y}(\cdot))$, let $\mathbf{s}_k \in \partial\phi^{CP}(\mathbf{x}, \boldsymbol{\xi}_k)$ be *any* subgradient of the fixed extension. The finite max-affine representation implies that

$$\mathbf{s}_k = \sum_{\ell \in A_k(\mathbf{x})} \theta_{k\ell} \mathbf{a}_{k\ell}, \qquad \theta_{k\ell} \geq 0, \qquad \sum_{\ell \in A_k(\mathbf{x})} \theta_{k\ell} = 1.$$

Accordingly, the convexified value-function constraint is

$$g_k(\mathbf{v}; \mathbf{w}) = q^{CP}(\mathbf{v}_y(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k) - \sum_{\ell \in A_k(\mathbf{x})} \theta_{k\ell}(\mathbf{a}_{k\ell}^\top \mathbf{v}_x + b_{k\ell}),$$

which is affine in $\mathbf{v}$ for the fixed base point and subgradient, without requiring that the selected subgradient be an extreme point of the subdifferential.

For each tuple of nonempty subsets $\mathcal{A} = (A_1, \dots, A_{N_{\boldsymbol{\xi}}})$, where $A_k \subseteq J_k$, define the polyhedron

$$\mathcal{P}_{\mathcal{A}} := \Big\{ \mathbf{v} \in \mathcal{W}^{CP} : C^{CP}(\mathbf{v}) - c \le 0, \quad H^{CP}_{i,k}(\mathbf{v}) \le 0 \ \ \forall i,k, \\ q^{CP}(\mathbf{v}_y(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k) - \mathbf{a}_{k\ell}^\top \mathbf{v}_x - b_{k\ell} \le 0 \ \ \forall k, \ \ell \in A_k \Big\}.$$

There are finitely many such polyhedra. For every empty $\mathcal{P}_{\mathcal{A}}$, compactness of $\mathcal{W}^{CP}$ implies that the minimum, over $\mathbf{v} \in \mathcal{W}^{CP}$, of the maximum of its defining constraint violations is strictly positive. Consequently, there exists $\rho > 0$ such that every subset tuple whose defining inequalities admit a point in $\mathcal{W}^{CP}$ with violations strictly smaller than ρ has $\mathcal{P}_{\mathcal{A}} \neq \varnothing$. (If all these polyhedra are nonempty, any $\rho > 0$ can be used.)

Choose $\bar{\delta}_{CP} > 0$ so that $\delta + 4\delta^2 < \rho$ and $\delta \le 1/2$ whenever $0 < \delta \le \bar{\delta}_{CP}$. Fix such a δ and set $\epsilon_\delta = 2\delta$. For any $\mathbf{w} \in \mathcal{W}^{CP}$ satisfying (66), that is, $f_i(\mathbf{w}) \le \delta + 4\delta^2$ for every explicit DC constraint, the active subset tuple $\mathcal{A} = (A_k(\mathbf{x}))_k$ satisfies the defining inequalities of $\mathcal{P}_{\mathcal{A}}$ with violations at most $\delta + 4\delta^2$: every $\ell \in A_k(\mathbf{x})$ has

$$q^{CP}(\mathbf{y}(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k) - \mathbf{a}_{k\ell}^\top \mathbf{x} - b_{k\ell} = q^{CP}(\mathbf{y}(\boldsymbol{\xi}_k), \boldsymbol{\xi}_k) - \phi^{CP}(\mathbf{x}, \boldsymbol{\xi}_k).$$

Thus $\mathcal{P}_{\mathcal{A}}$ is nonempty. Choose $\mathbf{z} \in \mathcal{P}_{\mathcal{A}}$. Every defining affine-piece inequality is satisfied at $\mathbf{z}$; taking convex combinations with the coefficients $\theta_{k\ell}$ yields

$$g_i(\mathbf{z}; \mathbf{w}) \le 0, \qquad i \in [m].$$

Let $\mathbf{u} \in \operatorname{int}(\mathcal{W}^{CP})$ be fixed. Since there are finitely many affine pieces and all their convex combinations have bounded coefficients, there exists $R \ge 1$, independent of $\mathbf{w}$ and the selected subgradients, such that $g_i(\mathbf{u}; \mathbf{w}) \le R$ for all i. Choose $t = \delta/(4R)$ and set $\mathbf{v} = (1-t)\mathbf{z} + t\mathbf{u}$. Then $\mathbf{v} \in \operatorname{int}(\mathcal{W}^{CP})$ and, by affinity of the selected constraints,

$$g_i(\mathbf{v}; \mathbf{w}) \le (1-t) g_i(\mathbf{z}; \mathbf{w}) + tR \le \frac{\delta}{4}, \qquad i \in [m].$$

It follows that

$$g_i(\mathbf{v}; \mathbf{w}) - \delta \le -\frac{3\delta}{4} < -\frac{\delta}{2} = -\nu_\delta, \qquad i \in [m].$$

Hence Assumption 23B holds with $\nu_\delta = \delta/2$, uniformly over all base points satisfying (66) and all valid subgradient choices. □

C.2 Newsvendor pricing game

Consider (13) and set $\mathcal{W}^{PG} := \mathcal{X} \times \mathcal{Y}$, with the price and order-quantity bounds introduced in Section 2.3. Write $\mathcal{X}_i := [\underline{x}_i, \overline{x}_i]$ and $\mathcal{Y}_i := [\underline{y}_i, \overline{y}_i]$, so that $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ and $\mathcal{Y} = \prod_{i=1}^n \mathcal{Y}_i$. Let $\mathcal{D}_{\boldsymbol{\zeta}} = \{\boldsymbol{\zeta}_\ell\}_{\ell=1}^{N_{\boldsymbol{\zeta}}}$ denote the empirical demand scenarios, where $\boldsymbol{\zeta}_\ell = (\zeta_{\ell,i})_{i=1}^n$ and $\zeta_{\ell,i}$ is retailer i's demand in scenario ℓ. Define

$$L_i(y_i) := \frac{1}{N_{\boldsymbol{\zeta}}} \sum_{\ell=1}^{N_{\boldsymbol{\zeta}}} \left[b_i(\zeta_{\ell,i} - y_i)_+ + (h_i + r_i)(y_i - \zeta_{\ell,i})_+ \right], \tag{72}$$

$$q^{PG}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n \left[L_i(y_i) + (x_i - r_i) y_i \right], \qquad \phi^{PG}(\mathbf{x}) := \min_{\mathbf{y} \in \mathcal{Y}} q^{PG}(\mathbf{x}, \mathbf{y}). \tag{73}$$

For each retailer, write

$$q_i^{PG}(x_i, y_i) := L_i(y_i) + (x_i - r_i)y_i, \quad \phi_i^{PG}(x_i) := \min_{y_i \in \mathcal{Y}_i} q_i^{PG}(x_i, y_i), \quad F_i^{PG} := q_i^{PG} - \phi_i^{PG}.$$

The two explicit DC constraints are

$$F^{PG}(\mathbf{x}, \mathbf{y}) := q^{PG}(\mathbf{x}, \mathbf{y}) - \phi^{PG}(\mathbf{x}) = \sum_{i=1}^{n} F_i^{PG}(x_i, y_i) \le 0,$$

and

$$B^{PG}(\mathbf{x}, \mathbf{y}) := C^{PG}(\mathbf{x}, \mathbf{y}) - c \le 0, \qquad C^{PG}(\mathbf{x}, \mathbf{y}) := (\mathbf{c} - \mathbf{x})^\top \mathbf{y}.$$

Let

$$\mathcal{G}_i^{PG} := \left\{ (x_i, y_i) \in \mathcal{X}_i \times \mathcal{Y}_i : y_i \in \arg\min_{z_i \in \mathcal{Y}_i} q_i^{PG}(x_i, z_i) \right\}, \qquad \mathcal{G}^{PG} := \prod_{i=1}^{n} \mathcal{G}_i^{PG}.$$

The empirical quantile formula following (64) implies that $\mathcal{G}_i^{PG}$ is a finite union of horizontal and vertical segments. Its price breakpoints are

$$\tau_{i,j} := b_i + r_i - \frac{j}{N_\zeta}(b_i + h_i + r_i), \qquad j = 0, \dots, N_\zeta.$$

We say that the price and order-quantity boxes are *quantile-compatible* if, for every i, $\mathcal{G}_i^{PG}$ contains a nonincreasing polygonal path connecting $(\underline{x}_i, \overline{y}_i)$ to $(\overline{x}_i, \underline{y}_i)$. For instance, this holds when $\mathcal{X}_i = [-h_i, b_i + r_i]$ and $\mathcal{Y}_i = [\min_\ell \zeta_{\ell,i}, \max_\ell \zeta_{\ell,i}]$ with distinct endpoints. Under this condition, every diagonal line $\{(x_i + t, y_i + t) : t \in \mathbb{R}\}$ through $\mathcal{X}_i \times \mathcal{Y}_i$ intersects $\mathcal{G}_i^{PG}$. Consequently, $\mathcal{G}^{PG}$ is a finite union of polytopes, with at most one free coordinate in each pair (x_i, y_i) on any constituent polytope. In particular, C^{PG} is affine on each such polytope. Let $\mathcal{V}^{PG}$ denote the union of their vertex sets and define

$$\mathcal{K}^{PG} := \left\{ C^{PG}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}^{PG} \right\}. \tag{74}$$

Proposition 30 (Verification for the pricing game)**.** *Suppose the price and order-quantity boxes of Section 2.3 are nondegenerate and quantile-compatible, and*

$$c > c_{PG}^*, \qquad c \notin \mathcal{K}^{PG},$$

where c_{PG}^* is the optimal value of the empirical fairness-unaware pricing game (10). Using the DC decompositions specified below, Assumption 23A–E holds. In particular, there exist constants $\bar{\delta}_{PG} > 0$ and $\kappa_{PG} > 0$ such that condition B holds with $\epsilon_\delta = 2\delta$ and $\nu_\delta = \kappa_{PG}\delta$ for all $\delta \in (0, \bar{\delta}_{PG}]$.

Proof. Assumption 23A follows from the common quadratic shift. The set $\mathcal{W}^{PG}$ is convex and compact. Each L_i is convex piecewise linear, ϕ^{PG} is finite and concave piecewise affine on $\mathcal{X}$, and the convex components of the empirical fairness objective are Lipschitz on $\mathcal{W}^{PG}$ by Remark 15. This verifies Assumptions 23C–D. An optimal fairness-unaware pricing solution is follower-optimal, has cost $c_{PG}^* < c$, and provides the feasible initial point required by Assumption 23E.

For Assumption 23B, use the following convex components:

$$\begin{aligned}
F^{PG,+}(\mathbf{x}, \mathbf{y}) &:= \sum_{i=1}^{n} [L_i(y_i) - r_i y_i] - \phi^{PG}(\mathbf{x}) + \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2, \\
F^{PG,-}(\mathbf{x}, \mathbf{y}) &:= \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2, \\
B^{PG,+}(\mathbf{x}, \mathbf{y}) &:= \mathbf{c}^\top \mathbf{y} - c + \frac{1}{4}\|\mathbf{x} - \mathbf{y}\|^2, \\
B^{PG,-}(\mathbf{x}, \mathbf{y}) &:= \frac{1}{4}\|\mathbf{x} + \mathbf{y}\|^2.
\end{aligned}$$

These are valid DC decompositions because ϕ^{PG} is concave and $\frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2 = \mathbf{x}^\top\mathbf{y}$. At $\mathbf{w} = (\mathbf{x}, \mathbf{y})$, the corresponding convexifications at $\mathbf{v} = (\mathbf{x}', \mathbf{y}')$ are

$$g_F(\mathbf{v}; \mathbf{w}) = F^{PG}(\mathbf{x}', \mathbf{y}') + \frac{1}{4}\|(\mathbf{x}' - \mathbf{x}) - (\mathbf{y}' - \mathbf{y})\|^2, \tag{75}$$

$$g_B(\mathbf{v}; \mathbf{w}) = B^{PG}(\mathbf{x}', \mathbf{y}') + \frac{1}{4}\|(\mathbf{x}' - \mathbf{x}) + (\mathbf{y}' - \mathbf{y})\|^2. \tag{76}$$

Step 1: A follower-optimal reference point. Fix $\delta > 0$ sufficiently small and set $\epsilon_\delta = 2\delta$. Consider any $\mathbf{w} = (\mathbf{x}, \mathbf{y}) \in \mathcal{W}^{PG}$ satisfying

$$F^{PG}(\mathbf{w}) \le \delta + 4\delta^2, \qquad B^{PG}(\mathbf{w}) \le \delta + 4\delta^2.$$

By quantile compatibility, the line $(x_i + t_i, y_i + t_i)$ intersects $\mathcal{G}_i^{PG}$ for some $t_i \in \mathbb{R}$. Consequently, there exists

$$\mathbf{z} = (\mathbf{x} + \mathbf{t}, \mathbf{y} + \mathbf{t}) \in \mathcal{G}^{PG}, \qquad \mathbf{t} = (t_1, \dots, t_n)^\top.$$

The follower gap and the quadratic term in (75) both vanish at $\mathbf{z}$, so $g_F(\mathbf{z}; \mathbf{w}) = 0$. Moreover,

$$\|\mathbf{z} - \mathbf{w}\|^2 \le 2F^{PG}(\mathbf{w}). \tag{77}$$

Indeed, since $y_i + t_i$ is optimal at $x_i + t_i$,

$$\begin{aligned} F_i^{PG}(x_i, y_i) &\ge q_i^{PG}(x_i, y_i) - q_i^{PG}(x_i, y_i + t_i) \\ &= q_i^{PG}(x_i + t_i, y_i) - q_i^{PG}(x_i + t_i, y_i + t_i) + t_i^2 \\ &\ge t_i^2. \end{aligned}$$

Summing over i proves (77). In particular, $\|\mathbf{z} - \mathbf{w}\| \le C_R\sqrt{\delta}$ for a constant C_R independent of $\mathbf{w}$.

Step 2: Create a linear follower margin. Choose a fixed $t \in (0, 1)$, to be specified below, and set

$$\mathbf{u} := (1 - t)\mathbf{w} + t\mathbf{z}.$$

By convexity of $g_F(\cdot; \mathbf{w})$,

$$g_F(\mathbf{u}; \mathbf{w}) \le (1 - t)F^{PG}(\mathbf{w}) \le (1 - t)(\delta + 4\delta^2). \tag{78}$$

Also $\|\mathbf{u} - \mathbf{w}\| = t\|\mathbf{z} - \mathbf{w}\|$. Let L_B be a Lipschitz constant for B^{PG} on $\mathcal{W}^{PG}$. Equation (76) gives

$$g_B(\mathbf{u}; \mathbf{w}) \le \delta + 4\delta^2 + L_B t\|\mathbf{z} - \mathbf{w}\| + \frac{1}{2}t^2\|\mathbf{z} - \mathbf{w}\|^2. \tag{79}$$

Step 3: Correct the budget while preserving the linear follower margin. Since $c \notin \mathcal{K}^{PG}$,

$$\Delta := \min_{\bar{\mathbf{z}} \in \mathcal{V}^{PG}} |B^{PG}(\bar{\mathbf{z}})| > 0.$$

If $B^{PG}(\mathbf{z}) < -\Delta/4$, then $\|\mathbf{z} - \mathbf{w}\| = O(\sqrt{\delta})$ implies that $B^{PG}(\mathbf{w}) < -\Delta/8$ for all sufficiently small δ. Thus $g_B(\mathbf{u}; \mathbf{w})$ already has a fixed negative margin, and we may set $\mathbf{v} = \mathbf{u}$.

Otherwise, $B^{PG}(\mathbf{w}) \le \delta + 4\delta^2$ and (77) imply $|B^{PG}(\mathbf{z})| \le \Delta/4$ for sufficiently small δ. Choose a polyhedral cell $Q \subseteq \mathcal{G}^{PG}$ containing $\mathbf{z}$ and round each free coordinate of $\mathbf{z}$ to its nearest endpoint, obtaining a vertex $\bar{\mathbf{z}}$ of Q. Then

$$|B^{PG}(\bar{\mathbf{z}}) - B^{PG}(\mathbf{z})| \ge \frac{3\Delta}{4}.$$

Because B^{PG} is affine on Q and Q has at most n free coordinates, there exist constants $\beta, \ell > 0$, independent of $\mathbf{z}$ and δ, and a free coordinate of Q such that its budget slope has magnitude at least β and $\mathbf{z}$ is at distance at least ℓ from both endpoints of that coordinate's segment. The uniform constants exist because there are finitely many bounded graph cells. Indeed, if a_j is the budget slope along free coordinate j and d_j is its distance to the nearest endpoint, the preceding inequality gives $\sum_j |a_j| d_j \geq 3\Delta/4$. Hence some j satisfies $|a_j| d_j \geq 3\Delta/(4n)$. Uniform upper bounds on segment lengths and on $|a_j|$ then yield the stated positive lower bounds β and ℓ.

Let $\mathbf{d}$ be the signed unit direction along this coordinate that decreases the budget. For sufficiently small δ, the directional budget slope at points near $\mathbf{z}$ satisfies

$$D_{\mathbf{d}} B^{PG}(\mathbf{u}) \leq -\beta/2. \tag{80}$$

The quantile structure also yields a constant $C_0 > 0$, independent of $\mathbf{w}$, such that

$$|D_{\mathbf{d}} F^{PG}(\mathbf{u} + s\mathbf{d})| \leq C_0 F^{PG}(\mathbf{w}) \tag{81}$$

for all sufficiently small $s \geq 0$ that keep the selected coordinate within its graph segment. To verify this bound, first suppose the free coordinate is a price x_i on a horizontal segment with optimal order z_{y_i}. Since z_{x_i} is at least ℓ from neighboring price breakpoints, the retailer gap at the base point satisfies

$$F_i^{PG}(x_i, y_i) \geq \frac{\ell}{2} |y_i - z_{y_i}|.$$

Along the selected price direction, the directional derivative of the gap equals, up to sign, $u_{y_i} - z_{y_i}$. If the free coordinate is an order quantity y_i on a vertical segment with fixed price z_{x_i}, the analogous bound is

$$F_i^{PG}(x_i, y_i) \geq \frac{\ell}{2} |x_i - z_{x_i}|,$$

while the directional derivative equals, up to sign, $u_{x_i} - z_{x_i}$. Both cases establish (81), for example with $C_0 = 2/\ell$. The same argument applies to segments on the box boundary.

Fix $\kappa = t/16$. By (79), we can move from $\mathbf{u}$ along $\mathbf{d}$ by a distance

$$s = \frac{4}{\beta} \left[L_B t \|\mathbf{z} - \mathbf{w}\| + 4\delta^2 + \frac{1}{2} t^2 \|\mathbf{z} - \mathbf{w}\|^2 + 2\kappa\delta \right] = O(t\sqrt{\delta})$$

and obtain

$$g_B(\mathbf{u} + s\mathbf{d}; \mathbf{w}) \leq (1 - 2\kappa)\delta. \tag{82}$$

Indeed, the directional derivative of the budget surrogate is at most $-\beta/4$ along this segment for sufficiently small δ, because the quadratic correction has derivative $O(t\sqrt{\delta} + s)$. The chosen step lies within the segment for all sufficiently small δ.

Set $\mathbf{v} := \mathbf{u} + s\mathbf{d}$. By (81), the first-order increase in the follower gap is at most $C_0 F^{PG}(\mathbf{w}) s$. Moreover, because $\mathbf{z} - \mathbf{w} = (\mathbf{t}, \mathbf{t})$, the quadratic correction in (75) vanishes at $\mathbf{u}$; moving along a single price or quantity coordinate increases it by exactly $s^2/4$. Therefore,

$$\begin{aligned} g_F(\mathbf{v}; \mathbf{w}) &\leq (1 - t)(\delta + 4\delta^2) + C_0(\delta + 4\delta^2)s + \frac{s^2}{4} \\ &\leq (1 - t)\delta + O(\delta^2) + O(t\delta^{3/2}) + O(t^2\delta). \end{aligned}$$

More precisely, because $\kappa = t/16$, the bound $s \leq C_s t\sqrt{\delta}$ holds with C_s independent of $t, \delta, \mathbf{w}$ when t ranges over a fixed small interval and δ is sufficiently small relative to t. Thus the coefficient of the

$O(t^2\delta)$ term can be chosen independently of t. Choosing $t > 0$ small enough that the $O(t^2\delta)$ term is at most $t\delta/4$, and then choosing $\delta > 0$ sufficiently small, gives

$$g_F(\mathbf{v}; \mathbf{w}) \leq (1 - t/2)\delta \leq (1 - 2\kappa)\delta.$$

In the earlier case where the budget already had a fixed negative margin, the same follower bound follows directly from (78).

Finally, take a convex combination of $\mathbf{v}$ with a fixed point in $\mathrm{int}(\mathcal{W}^{PG})$, using a sufficiently small positive coefficient of order δ. Uniform Lipschitz continuity of the convexified constraints preserves the bounds

$$g_i(\mathbf{v}; \mathbf{w}) \leq (1 - \kappa)\delta, \qquad i \in [m],$$

where $\mathbf{v}$ now denotes the resulting interior point and $\kappa = t/16 > 0$ is independent of δ and $\mathbf{w}$. Consequently,

$$g_i(\mathbf{v}; \mathbf{w}) - \delta \leq -\kappa\delta < -\frac{\kappa\delta}{2}.$$

Thus Assumption 23B holds with $\nu_\delta = \kappa\delta/2$, that is, $\kappa_{PG} = \kappa/2$. □

Propositions 29 and 30 verify Assumption 23A–E for both applications. In both cases, for every sufficiently small algorithmic relaxation $\delta > 0$, the choice $\epsilon_\delta = 2\delta$ gives a uniform Slater margin of order δ, namely $\nu_\delta = \Theta(\delta)$. The admissible range can be taken as $\delta \in (0, \bar{\delta}]$, where $\bar{\delta} = \min\{\bar{\delta}_{CP}, \bar{\delta}_{PG}\}$.

D Proofs for Convergence Analysis

In this section, we present the technical proofs for the convergence of IDCA.

D.1 Proof of Lemma 24

Proof of Lemma 24. Assumption 23B implies that (42) satisfies Slater's condition. Hence, the KKT conditions (43)–(45) hold for some $\boldsymbol{\lambda} \in \mathbb{R}^m_+$. The stationarity condition (43) implies (34), while (46) ensures that $\|\widehat{\mathbf{w}} - \mathbf{w}\| \leq \epsilon$. Define

$$e_i := f_i(\mathbf{w}) - g_i(\widehat{\mathbf{w}}; \mathbf{w}) + \delta, \qquad i \in [m].$$

Then (44) and (45) imply $f_i(\mathbf{w}) \leq e_i,\ \lambda_i(f_i(\mathbf{w}) - e_i) = 0,\ i \in [m]$. By Assumption 23D, $g_i(\cdot; \mathbf{w})$ is $2M$-Lipschitz continuous on $\mathcal{W}$. Since $f_i(\mathbf{w}) = g_i(\mathbf{w}; \mathbf{w})$, we obtain

$$|e_i| = |g_i(\mathbf{w}; \mathbf{w}) - g_i(\widehat{\mathbf{w}}; \mathbf{w}) + \delta| \leq 2M\|\widehat{\mathbf{w}} - \mathbf{w}\| + |\delta| \leq \epsilon,$$

where the last inequality follows from (46) and $|\delta| \leq \epsilon/2$. Therefore, conditions (34)–(37) hold, and $\mathbf{w}$ is a nearly ϵ-KKT point of (33). □

D.2 Proof of Lemma 25

Proof of Lemma 25. Suppose $0 < \delta \leq \bar{\delta}$ and $f_i(\mathbf{w}) - \delta \leq \epsilon_\delta^2$ for all $i \in [m]$, Assumption 23B implies that (42) satisfies Slater's condition. Hence, the KKT conditions (43)–(45) hold.

Choose any $\widehat{\mathbf{f}}_i^- \in \partial f_i^-(\widehat{\mathbf{w}})$ for $i = 0, 1, \dots, m$. Since $\mathbf{n}_{\mathcal{W}} \in \mathcal{N}_{\mathcal{W}}(\widehat{\mathbf{w}})$, we have $\langle \mathbf{n}_{\mathcal{W}}, \mathbf{w} - \widehat{\mathbf{w}} \rangle \le 0$. Combining this inequality with (43) gives

$$\begin{aligned}
0 &\ge \left\langle \mathbf{f}_0^+ - \mathbf{f}_0^- + \sum_{i=1}^m \lambda_i (\mathbf{f}_i^+ - \mathbf{f}_i^-), \widehat{\mathbf{w}} - \mathbf{w} \right\rangle \\
&= \left\langle \mathbf{f}_0^+ - \widehat{\mathbf{f}}_0^- + \sum_{i=1}^m \lambda_i (\mathbf{f}_i^+ - \widehat{\mathbf{f}}_i^-), \widehat{\mathbf{w}} - \mathbf{w} \right\rangle + \left\langle \widehat{\mathbf{f}}_0^- - \mathbf{f}_0^- + \sum_{i=1}^m \lambda_i (\widehat{\mathbf{f}}_i^- - \mathbf{f}_i^-), \widehat{\mathbf{w}} - \mathbf{w} \right\rangle \\
&\ge -2 \left(1 + \sum_{i=1}^m \lambda_i \right) M \|\widehat{\mathbf{w}} - \mathbf{w}\| + \mu \|\widehat{\mathbf{w}} - \mathbf{w}\|^2,
\end{aligned}$$

where the last inequality follows from the Lipschitz continuity in Assumption 23D and the strong convexity of f_0^-. Rearranging yields the first inequality in (47).

Next, let $\mathbf{v} \in \operatorname{int}(\mathcal{W})$ be the point guaranteed by Assumption 23B, so that $g_i(\mathbf{v}; \mathbf{w}) \le \delta - \nu_\delta$ for all $i \in [m]$. Since $\mathbf{f}_i^+ - \mathbf{f}_i^- \in \partial g_i(\widehat{\mathbf{w}}; \mathbf{w})$, convexity gives

$$\delta - \nu_\delta \ge g_i(\mathbf{v}; \mathbf{w}) \ge g_i(\widehat{\mathbf{w}}; \mathbf{w}) + \left\langle \mathbf{f}_i^+ - \mathbf{f}_i^-, \mathbf{v} - \widehat{\mathbf{w}} \right\rangle.$$

Multiplying by λ_i and summing over $i \in [m]$, we obtain

$$\begin{aligned}
(\delta - \nu_\delta) \sum_{i=1}^m \lambda_i &\ge \sum_{i=1}^m \lambda_i g_i(\widehat{\mathbf{w}}; \mathbf{w}) + \left\langle \sum_{i=1}^m \lambda_i (\mathbf{f}_i^+ - \mathbf{f}_i^-), \mathbf{v} - \widehat{\mathbf{w}} \right\rangle \\
&\ge \sum_{i=1}^m \lambda_i g_i(\widehat{\mathbf{w}}; \mathbf{w}) + \left\langle \sum_{i=1}^m \lambda_i (\mathbf{f}_i^+ - \mathbf{f}_i^-) + \mathbf{n}_{\mathcal{W}}, \mathbf{v} - \widehat{\mathbf{w}} \right\rangle \\
&= \delta \sum_{i=1}^m \lambda_i - \left\langle \mathbf{f}_0^+ - \mathbf{f}_0^-, \mathbf{v} - \widehat{\mathbf{w}} \right\rangle \\
&\ge \delta \sum_{i=1}^m \lambda_i - 2MD.
\end{aligned}$$

Here, the second inequality follows from $\mathbf{n}_{\mathcal{W}} \in \mathcal{N}_{\mathcal{W}}(\widehat{\mathbf{w}})$, the equality follows from (43) and (45), and the last inequality follows from Assumptions 23C–D. Rearranging yields the multiplier bound in (47). □

D.3 Proof of Theorem 26

Proof of Theorem 26. Since $\mathbf{w}^{(0)}$ is feasible, we trivially have $f_i(\mathbf{w}^{(0)}) - \delta_0 \le -\delta_0 \le \epsilon_{\delta_0}^2$ for all $i \in [m]$. By (40) and (41), $f_i(\mathbf{w}^{(k+1)}) \le g_i(\mathbf{w}^{(k+1)}; \mathbf{w}^{(k)}) \le \delta_k + \epsilon_k$ for any $i \in [m]$. By the condition $\delta_k + \epsilon_k \le \delta_{k+1} + \epsilon_{\delta_{k+1}}^2$, it follows recursively that

$$f_i(\mathbf{w}^{(k+1)}) - \delta_{k+1} \le \delta_k + \epsilon_k - \delta_{k+1} \le \epsilon_{\delta_{k+1}}^2, \qquad i \in [m], \quad k = 0, \dots, K-1. \tag{83}$$

Thus, at every iteration, Slater's condition holds for (38), and by Lemma 25, $\sum_{i=1}^m \lambda_i^{(k)} \le \frac{2MD}{\nu_{\delta_k}}$.

Because $g_0(\cdot;\mathbf{w}^{(k)})$ is μ-strongly convex, the KKT conditions of (38) imply

$$\begin{aligned}
&f_0(\mathbf{w}^{(k)}) + \sum_{i=1}^m \lambda_i^{(k)} f_i(\mathbf{w}^{(k)}) \\
&= g_0(\mathbf{w}^{(k)};\mathbf{w}^{(k)}) + \sum_{i=1}^m \lambda_i^{(k)} (g_i(\mathbf{w}^{(k)};\mathbf{w}^{(k)}) - \delta_k) + \delta_k \sum_{i=1}^m \lambda_i^{(k)} \\
&\geq g_0(\widehat{\mathbf{w}}^{(k+1)};\mathbf{w}^{(k)}) + \sum_{i=1}^m \lambda_i^{(k)} (g_i(\widehat{\mathbf{w}}^{(k+1)};\mathbf{w}^{(k)}) - \delta_k) + \delta_k \sum_{i=1}^m \lambda_i^{(k)} + \frac{\mu}{2}\|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2 \\
&\geq g_0(\mathbf{w}^{(k+1)};\mathbf{w}^{(k)}) - \epsilon_k + \delta_k \sum_{i=1}^m \lambda_i^{(k)} + \frac{\mu}{2}\|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2 \\
&\geq f_0(\mathbf{w}^{(k+1)}) - \epsilon_k + \delta_k \sum_{i=1}^m \lambda_i^{(k)} + \frac{\mu}{2}\|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2.
\end{aligned}$$

Using again the fact that $f_i(\mathbf{w}^{(k+1)}) \leq \delta_k + \epsilon_k$ for $k = -1, 0, 1, \ldots$ with $\delta_{-1} = \epsilon_{-1} = 0$, the inequality above implies

$$f_0(\mathbf{w}^{(k)}) + (\delta_{k-1} + \epsilon_{k-1}) \sum_{i=1}^m \lambda_i^{(k)} \geq f_0(\mathbf{w}^{(k+1)}) - \epsilon_k + \delta_k \sum_{i=1}^m \lambda_i^{(k)} + \frac{\mu}{2}\|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2. \tag{84}$$

Summing (84) over $k = 0, \ldots, K-1$ gives

$$\begin{aligned}
f_0(\mathbf{w}^{(0)}) - f_{\text{lb}} &\geq -\sum_{k=0}^{K-1} \epsilon_k - \sum_{k=0}^{K-1} \left(\sum_{i=1}^m \lambda_i^{(k)} \right) (\delta_{k-1} + \epsilon_{k-1} - \delta_k) + \frac{\mu}{2} \sum_{k=0}^{K-1} \|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2 \\
&\geq -\sum_{k=0}^{K-1} \epsilon_k - \sum_{k=0}^{K-1} \frac{2MD(\delta_{k-1} + \epsilon_{k-1} - \delta_k)_+}{\nu_{\delta_k}} + \frac{\mu}{2} \sum_{k=0}^{K-1} \|\widehat{\mathbf{w}}^{(k+1)} - \mathbf{w}^{(k)}\|^2.
\end{aligned}$$

Dividing by K yields (48).

When $\delta_k = \delta_\epsilon$ and (49)–(50) hold for any $k \geq 0$, we have $\epsilon_k \leq \epsilon_{\delta_\epsilon}^2$ so (48) holds, which implies

$$\min_{0 \leq k' \leq K-1} \left[\|\widehat{\mathbf{w}}^{(k'+1)} - \mathbf{w}^{(k')}\| \right] \leq \min\left\{1, \frac{1}{4M}\right\} \epsilon.$$

Let k' be the index achieving the minimum on the left-hand side above. By (83) and Lemma 24, $\mathbf{w}^{(k')}$ is nearly ϵ-KKT point of (33). This proves that Algorithm 1 finds a nearly ϵ-KKT point within $K = O(\epsilon^{-2})$ outer iterations.

□

D.4 Proof of Theorem 27

Proof of Theorem 27. We prove this theorem by directly applying Theorem 1 in Boob et al. (2023) to Algorithm 2 after verifying the assumptions of Theorem 1 in Boob et al. (2023) hold for (38).

By Assumption 23D, we have $|f_i^+(\mathbf{v}) - f_i^+(\mathbf{z})| \leq M\|\mathbf{v} - \mathbf{z}\|$, $\|\mathbf{f}_i^-(\mathbf{w}^{(k)})\|_2 \leq M$, $\|\mathbf{f}_i^+(\mathbf{v})\|_2 \leq M$, for any $\mathbf{v}, \mathbf{z} \in \mathcal{W}$ and $i = 0, \ldots, m$. Then we have

$$\begin{aligned}
|g_i(\mathbf{v}) - g_i(\mathbf{z})| &= \left| f_i^+(\mathbf{v}) - f_i^+(\mathbf{z}) - \langle \mathbf{f}_i^-, \mathbf{v} - \mathbf{z} \rangle \right| \\
&\leq |f_i^+(\mathbf{v}) - f_i^+(\mathbf{z})| + \|\mathbf{f}_i^-\|_2 \|\mathbf{v} - \mathbf{z}\| \leq 2M\|\mathbf{v} - \mathbf{z}\|,
\end{aligned} \tag{85}$$

for all $\mathbf{v}, \mathbf{z} \in \mathcal{W}$, and $i = 0, \ldots, m$. Moreover, since $\mathbf{g}_i(\mathbf{z}) = \mathbf{f}_i^+(\mathbf{z}) - \mathbf{f}_i^-$, we have

$$\begin{aligned}|g_i(\mathbf{v}) - g_i(\mathbf{z}) - \langle \mathbf{g}_i(\mathbf{z}), \mathbf{v} - \mathbf{z}\rangle| &= |f_i^+(\mathbf{v}) - f_i^+(\mathbf{z}) - \langle \mathbf{f}_i^+(\mathbf{z}), \mathbf{v} - \mathbf{z}\rangle| \\ &\leq |f_i^+(\mathbf{v}) - f_i^+(\mathbf{z})| + \|\mathbf{f}_i^+(\mathbf{z})\|_2 \|\mathbf{v} - \mathbf{z}\| \leq 2M\|\mathbf{v} - \mathbf{z}\|.\end{aligned}$$

Therefore, the constants L_i and $M_{\chi,i}$ in Boob et al. (2023) equal zero, while the constants H_i and $M_{f,i}$ in Boob et al. (2023) equal $2M$ for (38) for $i = 0, 1, \ldots, m$. It is also easy to see that α_0 in Boob et al. (2023) equals μ by Assumption 23A, while σ_f and σ_i, $i = 0, 1, \ldots, m$, in Boob et al. (2023) equals zero because Algorithm 2 has access to the deterministic subgradients. Therefore, all assumptions of Theorem 1 in Boob et al. (2023) are satisfied. The conclusion of this theorem is just that of Theorem 1 when applied to to Algorithm 2 with the constants specified as above.

□

E Computable Lower Bounds

In this section, we develop computable lower bounds on the optimal objective values of (9) and (13). These lower bounds are based on Lemma 10 and are used to evaluate the quality of the solutions obtained by IDCA and the exact method.

E.1 A Lower Bound for the Capacity-Planning Problem

We first replace the objective of (9) with the Jensen lower bound established in Lemma 10. Define

$$\begin{aligned}\mathbb{U}_{LB}^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) &:= \max_{s,s' \in \mathcal{S}} \left\{ \mathbb{E}_{\mathbb{P}_s} \left[U_s(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) \right] - \mathbb{E}_{\mathbb{P}_{s'}} \left[U_{s'}(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta}) \right] \right\}^2 \\ &= \max_{s,s' \in \mathcal{S}} \left\{ \frac{1}{m_s} \sum_{j=1}^{m_s} \mathbb{E}_{\boldsymbol{\xi}} \left[\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi}) \right] - \frac{1}{m_{s'}} \sum_{j=1}^{m_{s'}} \mathbb{E}_{\boldsymbol{\xi}} \left[\xi_j^{s'} - \sum_{i=1}^{n} y_{ij}^{s'}(\boldsymbol{\xi}) \right] \right\}^2, \end{aligned} \tag{86}$$

where the second equality follows from the definition of the community utility in (8). Lemma 10 implies that $\mathbb{U}_{LB}^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) \leq \mathbb{U}^{CP}(\mathbf{x}, \mathbf{y}(\cdot))$ for every feasible solution.

We next construct an equivalent reformulation of the lower-level optimality constraints in (9). For a fixed $\mathbf{x}$ and scenario $\boldsymbol{\xi}$, let $\alpha_i(\boldsymbol{\xi})$ and $\beta_j^s(\boldsymbol{\xi})$ denote the nonnegative dual variables associated with the capacity and demand constraints in (7), respectively. By primal feasibility, dual feasibility, and strong duality, $\mathbf{y}(\boldsymbol{\xi})$ is an optimal second-stage solution if and only if there exist $\alpha_i(\boldsymbol{\xi}) \geq 0$ and $\beta_j^s(\boldsymbol{\xi}) \geq 0$ satisfying

$$\sum_{s \in \mathcal{S}} \sum_{j=1}^{m_s} y_{ij}^s(\boldsymbol{\xi}) \leq x_i o_i, \qquad i = 1, \ldots, n, \quad \boldsymbol{\xi} \in \Xi, \tag{87}$$

$$\sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi}) \leq \xi_j^s, \qquad s \in \mathcal{S}, \quad j = 1, \ldots, m_s, \quad \boldsymbol{\xi} \in \Xi, \tag{88}$$

$$c_{ij}^s - p_s + \alpha_i(\boldsymbol{\xi}) + \beta_j^s(\boldsymbol{\xi}) \geq 0, \qquad i = 1, \ldots, n, \quad s \in \mathcal{S}, \quad j = 1, \ldots, m_s, \quad \boldsymbol{\xi} \in \Xi, \tag{89}$$

together with

$$\mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{i=1}^{n}\sum_{s\in S}\sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi})\right] + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in S}\sum_{j=1}^{m_s} p_s\left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi})\right)\right]$$
$$\leq N_{\boldsymbol{\xi}}\mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in S}\sum_{j=1}^{m_s} p_s\xi_j^s - \sum_{i=1}^{n}\alpha_i(\boldsymbol{\xi})x_i o_i - \sum_{s\in S}\sum_{j=1}^{m_s}\beta_j^s(\boldsymbol{\xi})\xi_j^s\right]. \tag{90}$$

Weak duality gives the reverse inequality for every primal-dual feasible solution. Therefore, (90) enforces strong duality under the empirical distribution.

Replacing the original objective of (9) by (86) and using the equivalent lower-level optimality constraints above yield the following optimization problem whose optimal objective value is a lower bound of that of (9):

$$\begin{aligned}\min_{\substack{\mathbf{x}\in[0,1]^n,\ \mathbf{y}(\cdot)\geq 0,\\ \boldsymbol{\alpha}(\cdot)\geq 0,\ \boldsymbol{\beta}(\cdot)\geq 0}} \quad & \mathbb{U}_{LB}^{CP}(\mathbf{x},\mathbf{y}(\cdot)) \\ \text{s.t.} \quad & \mathbf{g}^\top\mathbf{x} + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi})\right] + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} p_s\left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi})\right)\right] \leq c, \\ & (87)\text{–}(90).\end{aligned} \tag{91}$$

Although (91) provides a valid lower bound, it remains nonconvex because (90) contains the bilinear terms $\alpha_i(\boldsymbol{\xi})x_i o_i$. To obtain a tractable lower bound, we further relax (87) to a linear inequality using the technique from Kleinert and Schmidt (2023). In particular, for each production site i, define

$$\begin{aligned} G_i^+ := \min_{\mathbf{x}\in[0,1]^n,\ \mathbf{y}(\cdot)\geq 0} \quad & x_i o_i \\ \text{s.t.} \quad & \sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} y_{\ell j}^s(\boldsymbol{\xi}) \leq x_\ell o_\ell, \quad \ell\in\mathcal{L}, \quad \boldsymbol{\xi}\in\Xi, \\ & \sum_{\ell=1}^{n} y_{\ell j}^s(\boldsymbol{\xi}) \leq \xi_j^s, \quad s\in\mathcal{S}, \quad j=1,\ldots,m_s, \quad \boldsymbol{\xi}\in\Xi, \\ & \mathbf{g}^\top\mathbf{x} + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi})\right] + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} p_s\left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi})\right)\right] \leq c. \end{aligned} \tag{92}$$

Thus, for every pair $(\mathbf{x},\mathbf{y}(\cdot))$ satisfying the primal feasibility and cost constraints, $G_i^+ \leq x_i o_i, \qquad i = 1,\ldots,n$. Because $\alpha_i(\boldsymbol{\xi}) \geq 0$, replacing $x_i o_i$ by G_i^+ on the right-hand side of (90) enlarges the feasible region. In particular, (90) can be relaxed to

$$\mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{i=1}^{n}\sum_{s\in S}\sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi})\right] + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in S}\sum_{j=1}^{m_s} p_s\left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi})\right)\right]$$
$$\leq N_{\boldsymbol{\xi}}\mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in S}\sum_{j=1}^{m_s} p_s\xi_j^s - \sum_{i=1}^{n}\alpha_i(\boldsymbol{\xi})G_i^+ - \sum_{s\in S}\sum_{j=1}^{m_s}\beta_j^s(\boldsymbol{\xi})\xi_j^s\right]. \tag{93}$$

All terms in (93) are linear in the decision variables because the values G_i^+ are computed in advance.

We therefore obtain the following convex lower-bounding problem:

$$\mathrm{LB}_{\mathrm{CP}} := \min_{\substack{\mathbf{x}\in[0,1]^n,\ \mathbf{y}(\cdot)\geq 0,\\ \boldsymbol{\alpha}(\cdot)\geq 0,\ \beta(\cdot)\geq 0}} \quad \mathbb{U}_{LB}^{CP}(\mathbf{x}, \mathbf{y}(\cdot)) \tag{94}$$

$$\text{s.t.} \quad \mathbf{g}^\top \mathbf{x} + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{i=1}^{n}\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} c_{ij}^s y_{ij}^s(\boldsymbol{\xi})\right] + \mathbb{E}_{\boldsymbol{\xi}}\left[\sum_{s\in\mathcal{S}}\sum_{j=1}^{m_s} p_s\left(\xi_j^s - \sum_{i=1}^{n} y_{ij}^s(\boldsymbol{\xi})\right)\right] \leq c,$$

$$(87)\text{–}(89), \quad (93).$$

The objective in (94) is the maximum of convex quadratic functions, and all its constraints are linear. Hence (94) is a convex optimization problem.

Finally, every feasible solution of (9) can be extended with lower-level dual solutions satisfying (87)–(90), and therefore also satisfies the relaxed constraint (93). Moreover, its objective value is bounded below by (86). Thus, $\mathrm{LB}_{\mathrm{CP}}$ is a valid lower bound on the optimal distributional-unfairness value of the capacity-planning problem.

F The MINLPs Solved by the Exact Methods

F.1 The Sorted Demand for Pricing Game Problem

Lemma 31. *Let $\zeta_{s,k}$ be the demand from community s in scenarion k for $s = 1,\dots,S$ and $k = 1,\dots,N_\zeta$. We assume $\zeta_{s,k}$'s are sorted across scenario for each s, namely, $\zeta_{s,1} \leq \zeta_{s,2} \leq \cdots \leq \zeta_{s,N_\zeta}$. Then, we can show that the Wasserstein distance between two communities can be computed through a one-to-one comparison of their utilities in each scenario. More specifically, we define*

$$\mathcal{F}'^{PG} = \left\{ (\mathbf{x}, \mathbf{y}, \nu, \bar{u}_{ks}) \in \mathcal{X}\times\mathcal{Y}\times\mathbb{R}_+ : \ \frac{1}{N}\sum_{k=1}^{N}(\bar{u}_{ks} - \bar{u}_{ks'})^2 \leq \nu, \ \forall s, s' \in \mathcal{S}. \right\}. \tag{95}$$

Then, (26) can be written as

$$\min_{\substack{(\mathbf{x},\mathbf{y},\nu,\bar{\mathbf{u}})\in\mathcal{F}'^{PG}\\ \boldsymbol{\lambda}(\cdot),\mathbf{w}(\cdot)}} \nu \ \ s.t.\ \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\zeta}}\left[G(\mathbf{x}, \mathbf{y}(\boldsymbol{\xi}), \boldsymbol{\xi}, \boldsymbol{\zeta})\right] \leq c, \ (23), (24), (25), \tag{96}$$

Proof. Since $\zeta_{s,1} \leq \cdots \leq \zeta_{s,N}$ for every $s \in \mathcal{S}$, the monotonicity of utility with respect to demand implies that the corresponding utility values are ordered in the same direction across communities. For any $s, s' \in \mathcal{S}$, the Wasserstein distance (15) can be written

$$\begin{aligned} \min_{\pi} \quad & \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_{ij}(\bar{u}_{is} - \bar{u}_{js'})^2 \\ \text{s.t.} \quad & \sum_{i=1}^{N} \pi_{ij} = \frac{1}{N}, \quad j = 1,\dots,N, \\ & \sum_{j=1}^{N} \pi_{ij} = \frac{1}{N}, \quad i = 1,\dots,N, \\ & \pi_{ij} \geq 0, \quad i,j = 1,\dots,N. \end{aligned} \tag{97}$$

Suppose, for contradiction, that the diagonal matching is not optimal, and consider an optimal plan. If $\bar{u}_{1s}$ is not fully matched with $\bar{u}_{1s'}$, the marginal constraints imply that some positive mass is

Table 4: Comparison among different reformulation for the exact problem

	Exact method based on (96)				Exact method based on (26)			
κ	UB by Gurobi	LB by Gurobi	Gurobi gap	Exact time	UB by Gurobi	LB by Gurobi	Gurobi gap	Exact time
0.70	1.50	1.50	0.00%	2598.19	NA	0.00	NA	TL
0.80	1.58	1.58	0.00%	817.37	NA	0.00	NA	TL
0.90	1.59	1.59	0.00%	761.56	NA	0.00	NA	TL
0.95	3.32	3.32	0.14%	TL	NA	0.00	NA	TL
0.96	3.86	3.86	0.00%	3626.07	NA	0.00	NA	TL
0.98	5.53	5.45	3.08%	TL	NA	0.00	NA	TL

matched between $\bar{u}_{1s}$ and $\bar{u}_{js'}$, and between $\bar{u}_{is}$ and $\bar{u}_{1s'}$, for some $i, j > 1$. We can instead match an equal amount of this mass between $\bar{u}_{1s}$ and $\bar{u}_{1s'}$, and between $\bar{u}_{is}$ and $\bar{u}_{js'}$. This preserves the marginal constraints. Since the utility values are ordered, $|\bar{u}_{1s}-\bar{u}_{1s'}|^2+|\bar{u}_{is}-\bar{u}_{js'}|^2 \le |\bar{u}_{1s}-\bar{u}_{js'}|^2+|\bar{u}_{is}-\bar{u}_{1s'}|^2$. Thus, this exchange does not increase the objective value in (97).

Repeat until $\bar{u}_{1s}$ is fully matched with $\bar{u}_{1s'}$, and apply the same argument to the remaining samples. This produces the diagonal matching with no greater cost, contradicting the assumption that it is not optimal. Since each sample has mass $1/N$, this plan satisfies $\pi^*_{ii} = 1/N$ and $\pi^*_{ij} = 0$ for $i \neq j$. Consequently, bounding this distance by ν is equivalent to imposing $\frac{1}{N}\sum_{k=1}^{N}(\bar{u}_{ks} - \bar{u}_{ks'})^2 \le \nu,\ \forall s, s' \in \mathcal{S}$. □

Since $\mathcal{F}'^{PG}$ is much simpler than $\mathcal{F}'$, Gurobi can solve (96) much more efficiently than (26). Table 4 compares the performance of Gurobi in solving these two formulations for each $\kappa \in \kappa^{PG}$. When solving either formulation, we impose a time limit of four hours (14,400 seconds). For each formulation, we report the objective value of (13) at termination, the lower bound produced by Gurobi, the optimality gap, and runtime. The optimality gap is computed as $\frac{\text{Obj}-\text{LB}}{\text{Obj}} \times 100\%$, where Obj denotes the reported objective value and $\text{LB} = \max\{\text{LB}_{\text{Gurobi1}}, \text{LB}_{\text{Gurobi2}}\}$. Here, $\text{LB}_{\text{Gurobi1}}$ and $\text{LB}_{\text{Gurobi2}}$ denote the lower bounds produced by Gurobi at termination when applied to (26) and (96), respectively, and "TL" indicates that the solver reaches the time limit. Gurobi solves (96) to global optimality (with a 0% optimality gap) within the time limit for most values of κ. For the remaining cases, the optimality gaps are all below 5%, indicating that Gurobi still achieves high-quality solutions. In contrast, Gurobi fails to find a feasible solution when applied to (26), so the objective value and optimality gap are reported as "NA".

F.2 A Subgradient-Based Reformulation of the Newsvendor Pricing Game

We derive an exact single-level formulation from the subgradient optimality conditions of the follower problem, eliminating the shortage and surplus variables used in the auxiliary-variable formulation.

Following Section C.2, let $\mathcal{L} := \{1, \ldots, N_\zeta\}$ and $\mathcal{D}_\zeta = \{\boldsymbol{\zeta}_\ell\}_{\ell \in \mathcal{L}}$, where $\boldsymbol{\zeta}_\ell = (\zeta_{\ell,i})_{i=1}^n$. Recall that $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ and $\mathcal{Y} = \prod_{i=1}^n \mathcal{Y}_i$, with $\mathcal{X}_i = [\underline{x}_i, \overline{x}_i]$ and $\mathcal{Y}_i = [\underline{y}_i, \overline{y}_i]$. All bounds are finite, $\underline{y}_i < \overline{y}_i$, $b_i > 0$, and $h_i + r_i > 0$ for each retailer i. Using L_i from (72), the empirical follower objective is

$$
\begin{aligned}
q^{PG}(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} \left[L_i(y_i) + (x_i - r_i) y_i \right] \\
&= \sum_{i=1}^{n} (x_i - r_i) y_i + \frac{1}{N_\zeta} \sum_{i=1}^{n} \sum_{\ell \in \mathcal{L}} \left[b_i (\zeta_{\ell,i} - y_i)_+ + (h_i + r_i)(y_i - \zeta_{\ell,i})_+ \right]. \qquad (98)
\end{aligned}
$$

This representation uses the identity $\min\{y_i, \zeta_{\ell,i}\} = y_i - (y_i - \zeta_{\ell,i})_+$. For fixed $\mathbf{x}$, $q^{PG}(\mathbf{x}, \cdot)$ is finite and convex.

Subgradient optimality conditions. For a fixed $\mathbf{x} \in \mathcal{X}$, a vector $\mathbf{y} \in \mathcal{Y}$ solves the follower problem if and only if

$$0 \in \partial_{\mathbf{y}} q^{PG}(\mathbf{x}, \mathbf{y}) + \mathcal{N}_{\mathcal{Y}}(\mathbf{y}),$$

where $\mathcal{N}_{\mathcal{Y}}(\mathbf{y})$ denotes the normal cone to $\mathcal{Y}$. For every retailer $i = 1, \ldots, n$ and scenario $\ell \in \mathcal{L}$, introduce continuous variables $\lambda^-_{i\ell}$ and $\lambda^+_{i\ell}$ satisfying

$$\lambda^-_{i\ell} \in \begin{cases} \{b_i\}, & y_i < \zeta_{\ell,i}, \\ [0, b_i], & y_i = \zeta_{\ell,i}, \\ \{0\}, & y_i > \zeta_{\ell,i}, \end{cases} \tag{99}$$

$$\lambda^+_{i\ell} \in \begin{cases} \{0\}, & y_i < \zeta_{\ell,i}, \\ [0, h_i + r_i], & y_i = \zeta_{\ell,i}, \\ \{h_i + r_i\}, & y_i > \zeta_{\ell,i}. \end{cases} \tag{100}$$

Indeed, $-\lambda^-_{i\ell}$ is a subgradient of $b_i(\zeta_{\ell,i} - y_i)_+$ with respect to y_i, and $\lambda^+_{i\ell}$ is a subgradient of $(h_i + r_i)(y_i - \zeta_{\ell,i})_+$. Thus, a subgradient of $q^{PG}(\mathbf{x}, \cdot)$ has ith component

$$x_i - r_i + \frac{1}{N_\zeta} \sum_{\ell \in \mathcal{L}} (\lambda^+_{i\ell} - \lambda^-_{i\ell}).$$

Introducing the nonnegative multipliers α_i and β_i associated with the lower and upper bounds on y_i, respectively, gives

$$x_i - r_i + \frac{1}{N_\zeta} \sum_{\ell \in \mathcal{L}} (\lambda^+_{i\ell} - \lambda^-_{i\ell}) - \alpha_i + \beta_i = 0, \tag{101}$$

$$\underline{y}_i \le y_i \le \overline{y}_i, \tag{102}$$

$$\alpha_i, \beta_i \ge 0,$$

$$\alpha_i (y_i - \underline{y}_i) = 0, \tag{103}$$

$$\beta_i (\overline{y}_i - y_i) = 0, \tag{104}$$

for $i = 1, \ldots, n$, together with (99) and (100). At a demand breakpoint $y_i = \zeta_{\ell,i}$, $\lambda^+_{i\ell} - \lambda^-_{i\ell}$ can take any value in $[-b_i, h_i + r_i]$; hence the full subdifferential at nondifferentiable points is retained.

Binary representation of the subgradient selections. For $i = 1, \ldots, n$ and $\ell \in \mathcal{L}$, define

$$B^-_{i\ell} := \max\{\zeta_{\ell,i} - \underline{y}_i, 0\}, \qquad B^+_{i\ell} := \max\{\overline{y}_i - \zeta_{\ell,i}, 0\}.$$

Introduce binary variables $u^-_{i\ell}, u^+_{i\ell}, v^-_{i\ell}, v^+_{i\ell}$. The shortage subgradient condition (99) is equivalent to

$$\begin{aligned} \zeta_{\ell,i} - y_i &\le B^-_{i\ell} u^-_{i\ell}, \\ 0 \le b_i - \lambda^-_{i\ell} &\le b_i (1 - u^-_{i\ell}), \\ y_i - \zeta_{\ell,i} &\le B^+_{i\ell} (1 - v^-_{i\ell}), \\ 0 \le \lambda^-_{i\ell} &\le b_i v^-_{i\ell}, \\ u^-_{i\ell}, v^-_{i\ell} &\in \{0, 1\}. \end{aligned} \tag{105}$$

If $y_i < \zeta_{\ell,i}$, the first inequality forces $u_{i\ell}^- = 1$, so $\lambda_{i\ell}^- = b_i$. If $y_i > \zeta_{\ell,i}$, the third inequality forces $v_{i\ell}^- = 0$, so $\lambda_{i\ell}^- = 0$. At equality, $u_{i\ell}^- = 0$ and $v_{i\ell}^- = 1$ permit any $\lambda_{i\ell}^- \in [0, b_i]$.

Similarly, (100) is equivalent to

$$\begin{aligned} y_i - \zeta_{\ell,i} &\le B_{i\ell}^+ u_{i\ell}^+, \\ 0 \le h_i + r_i - \lambda_{i\ell}^+ &\le (h_i + r_i)(1 - u_{i\ell}^+), \\ \zeta_{\ell,i} - y_i &\le B_{i\ell}^-(1 - v_{i\ell}^+), \\ 0 \le \lambda_{i\ell}^+ &\le (h_i + r_i) v_{i\ell}^+, \\ u_{i\ell}^+, v_{i\ell}^+ &\in \{0,1\}. \end{aligned} \tag{106}$$

The box complementary-slackness conditions have the same binary representation as in the auxiliary-variable formulation. Define

$$A_i := \max\{\overline{x}_i + h_i, 0\}, \qquad B_i := \max\{r_i - \underline{x}_i + b_i, 0\}.$$

Using binary variables $a_i^{\mathrm{L}}, a_i^{\mathrm{U}}$, impose

$$\begin{aligned} 0 \le \alpha_i &\le A_i a_i^{\mathrm{L}}, \\ 0 \le y_i - \underline{y}_i &\le (\overline{y}_i - \underline{y}_i)(1 - a_i^{\mathrm{L}}), \\ 0 \le \beta_i &\le B_i a_i^{\mathrm{U}}, \\ 0 \le \overline{y}_i - y_i &\le (\overline{y}_i - \underline{y}_i)(1 - a_i^{\mathrm{U}}), \\ a_i^{\mathrm{L}}, a_i^{\mathrm{U}} &\in \{0,1\}, \qquad i = 1, \dots, n. \end{aligned} \tag{107}$$

Equivalence to the auxiliary-variable formulation. Consider the epigraph formulation of the follower problem, using the scenario indexing $\ell \in \mathcal{L}$:

$$\zeta_{\ell,i} - y_i \le s_{i\ell}^-, \qquad y_i - \zeta_{\ell,i} \le s_{i\ell}^+, \qquad s_{i\ell}^-, s_{i\ell}^+ \ge 0.$$

Associate the multipliers $\lambda_{i\ell}^-/N_\zeta$ and $\lambda_{i\ell}^+/N_\zeta$ with the first and second epigraph inequalities, respectively. Stationarity with respect to $s_{i\ell}^-$ and $s_{i\ell}^+$ gives

$$0 \le \lambda_{i\ell}^- \le b_i, \qquad 0 \le \lambda_{i\ell}^+ \le h_i + r_i,$$

and complementary slackness gives

$$\begin{aligned} \lambda_{i\ell}^-(s_{i\ell}^- - \zeta_{\ell,i} + y_i) &= 0, & s_{i\ell}^-(b_i - \lambda_{i\ell}^-) &= 0, \\ \lambda_{i\ell}^+(s_{i\ell}^+ - y_i + \zeta_{\ell,i}) &= 0, & s_{i\ell}^+(h_i + r_i - \lambda_{i\ell}^+) &= 0. \end{aligned}$$

Because $b_i > 0$ and $h_i + r_i > 0$, every optimal solution satisfies

$$s_{i\ell}^- = (\zeta_{\ell,i} - y_i)_+, \qquad s_{i\ell}^+ = (y_i - \zeta_{\ell,i})_+.$$

Substitution yields exactly (99) and (100).

The binary constraints are equivalent as well. For example, $s_{i\ell}^- \le B_{i\ell}^- u_{i\ell}^-$ reduces to $\zeta_{\ell,i} - y_i \le B_{i\ell}^- u_{i\ell}^-$, and $s_{i\ell}^- - \zeta_{\ell,i} + y_i \le B_{i\ell}^+(1 - v_{i\ell}^-)$ reduces to $y_i - \zeta_{\ell,i} \le B_{i\ell}^+(1 - v_{i\ell}^-)$. The surplus constraints follow analogously. Conversely, any feasible point of (105) and (106) extends to a feasible point of the auxiliary-variable formulation by the positive-part assignments above. Hence the direct subgradient formulation is precisely the projection of the auxiliary-variable formulation obtained by eliminating $\mathbf{s}^-$ and $\mathbf{s}^+$. The box multipliers and their binary representations remain unchanged. (In the auxiliary-variable formulation, the scenario index k is the index ℓ used here.)

The resulting single-level formulation. Let $\mathcal{F}'^{PG}$ denote the sorted-demand Wasserstein epigraph set defined in Section F.1, with auxiliary utility variables $\bar{\mathbf{u}} = (\bar{u}_{\ell,s})_{\ell\in\mathcal{L},\, s\in\mathcal{S}}$. Here $\zeta_{(\ell),s}$ denotes the ℓth smallest demand of community s, and the corresponding sorted utility is $\bar{u}_{\ell,s} = (\zeta_{(\ell),s} - y_s)_+$. The empirical distributionally fair newsvendor pricing game admits the equivalent formulation

$$\begin{aligned} \min_{\substack{\mathbf{x},\mathbf{y},\boldsymbol{\lambda}^-,\boldsymbol{\lambda}^+,\boldsymbol{\alpha},\boldsymbol{\beta},\\ \mathbf{u}^-,\mathbf{u}^+,\mathbf{v}^-,\mathbf{v}^+,\\ \mathbf{a}^{\mathrm{L}},\mathbf{a}^{\mathrm{U}},\nu,\bar{\mathbf{u}}}} \quad & \nu \\ \text{s.t.} \quad & (\mathbf{x},\mathbf{y},\nu,\bar{\mathbf{u}}) \in \mathcal{F}'^{PG}, \\ & \bar{u}_{\ell,s} = (\zeta_{(\ell),s} - y_s)_+, \quad \ell\in\mathcal{L},\ s\in\mathcal{S}, \\ & (\mathbf{c}-\mathbf{x})^\top \mathbf{y} \le c, \\ & \mathbf{x}\in\mathcal{X}, \\ & (101), \\ & (105), \quad (106), \quad (107). \end{aligned} \tag{108}$$

The lower-level optimality conditions in this formulation are mixed-integer linear constraints. Its nonlinearities arise from the leader's bilinear budget constraint and the representation of the fairness epigraph.

G Post Process by Projection for Newsvendor Pricing Game

Starting from the solution returned by Algorithm 1, we apply alternating projections onto the revenue constraint set and the bilevel feasible set to recover feasibility for (13). The revenue projection is computed by bisection, while the bilevel projection compares the nearest points on the horizontal and vertical segments of the lower-level solution graph. Algorithm 3 summarizes this procedure.

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Algorithm 3 Alternating Projection for Feasibility Recovery

Input: The solution $(\mathbf{x}^{(k')}, \mathbf{y}^{(k')})$ from Algorithm 1, the pricing-game data, and maximum outer iterations $T_p = 100{,}000$.
Initialize $(\mathbf{x}, \mathbf{y}) \leftarrow (\mathbf{x}^{(k')}, \mathbf{y}^{(k')})$.
for $t = 1, \ldots, T_p$ **do**
  Revenue projection: Set $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \leftarrow (\mathbf{x}, \mathbf{y})$.
  if $\sum_i (c_i - \widehat{x}_i)\widehat{y}_i > c$ **then**
    For $\lambda \in [0, 1)$, compute

$$y_i(\lambda) = \frac{\widehat{y}_i + \lambda(\widehat{x}_i - c_i)}{1 - \lambda^2}, \qquad x_i(\lambda) = \widehat{x}_i + \lambda y_i(\lambda), \quad i = 1, \ldots, n.$$

    Set $\lambda_{\text{lo}} = 0$ and choose $\lambda_{\text{hi}} < 1$ such that $\sum_i (c_i - x_i(\lambda_{\text{hi}}))y_i(\lambda_{\text{hi}}) \le c$.
    repeat
      Set $\lambda = (\lambda_{\text{lo}} + \lambda_{\text{hi}})/2$ and compute $(\mathbf{x}(\lambda), \mathbf{y}(\lambda))$.
      If $\sum_i (c_i - x_i(\lambda))y_i(\lambda) > c$, set $\lambda_{\text{lo}} \leftarrow \lambda$; otherwise set $\lambda_{\text{hi}} \leftarrow \lambda$.
    until $\sum_i (c_i - x_i(\lambda))y_i(\lambda) + 0.0001 \le c$
    Set $(\mathbf{x}, \mathbf{y}) \leftarrow (\mathbf{x}(\lambda), \mathbf{y}(\lambda))$.
  end if
  Return $(\mathbf{x}, \mathbf{y})$ if feasible for (13) within 0.0001.
  Bilevel projection: Set $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \leftarrow (\mathbf{x}, \mathbf{y})$.
  for $i = 1, \ldots, n$ **do**
    For each $k = 1, \ldots, m$, form a candidate (x_i, ξ_{ki}) using the interval

$$\left[b_i + r_i - \frac{k}{m}(b_i + r_i + h_i),\ b_i + r_i - \frac{k-1}{m}(b_i + r_i + h_i)\right].$$

    Keep $x_i = \widehat{x}_i$ if $\widehat{x}_i$ lies in this interval; otherwise choose the endpoint closest to $\widehat{x}_i$.
    For each $k = 1, \ldots, m-1$, form a candidate $\left(b_i + r_i - \frac{k}{m}(b_i + r_i + h_i),\ y_i\right)$.
    Keep $y_i = \widehat{y}_i$ if $\xi_{ki} \le \widehat{y}_i \le \xi_{k+1,i}$; otherwise choose the endpoint ξ_{ki} or $\xi_{k+1,i}$ closest to $\widehat{y}_i$.
    Select (x_i, y_i) among all candidates to minimize $(x_i - \widehat{x}_i)^2 + (y_i - \widehat{y}_i)^2$.
  end for
  Return $(\mathbf{x}, \mathbf{y})$ if feasible for (13) within 0.0001.
end for
Output: The final point $(\mathbf{x}, \mathbf{y})$.

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